1-1-2016

Optimal Control Of A Perturbed Sweeping Process With Applications To The Crowd Motion Model

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OPTIMAL CONTROL OF A PERTURBED SWEEPING PROCESS WITH APPLICATIONS TO THE CROWD MOTION MODEL

by

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DISSERTATION

Submitted to the Graduate School

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2016

MAJOR: (APPLIED) MATHEMATICS

Approved By:

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Advisor Date

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DEDICATION

To my father, mother, and my brother

Cao Hoàng Tấn, Nguyễn Thị Kim Sang, and Cao Hoàng Thắng

ACKNOWLEDGEMENTS

Though only my name appears on the cover of this dissertation, a great many people have contributed to its production, I owe my gratitude to all those people who have made this dissertation possible and because of whom my graduate experience has been one that I will cherish forever.

First and foremost I would like to express my deepest gratitude to my advisor Professor Boris Mordukhovich for his endless support of my Ph.D study and related research at Wayne State University. His enthusiasm, dedication, along with his immense knowledge in the theory of variational analysis, generalized differentiations, and optimal control did encourage me to overcome a large number of principal and technical difficulties in my research. Needless to say, his guidance and patience helped me in all the time of research and writing of this dissertation. I could not have imagined having a better advisor and mentor for my Ph.D study. It is a great pleasure to study with him and particularly have him as my advisor during 5 years at Wayne State University.

Besides my advisor, I would like to thank the rest of my thesis committee: Professor Jose Luis Menaldi, Professor Alper Murat, Professor Peiyong Wang, and Professor George Yin for their encouragment and insightful comments. I am particularly grateful to Professor Giovanni Colombo, Dr. Hoang Nguyen for helpful discussions on discrete approximations of non-Lipschitzian differential inclusions, and Dr. Juliette Venel for her remarkable discussions on the crowd motion model in a corridor and its applications.

My sincere thanks also goes to Ms. Mary Klamo, Ms. Patricia Bonesteel, Ms. Tiana Fluker, Ms. Barbara Malicke, Ms. Shereen Schultz, Mr. Christopher Leirstein, Professor

Daniel Frohardt, Professor Robert Bruner, and the entire Department of Mathematics for their kind support in a number of ways.

I greatly appreciate the financial support from the National Science Foundation and the Air Force Office of Scientific Research that funded parts of my research in this dissertation.

Many friends of mine have helped me stay sane through these difficult years. Their support and care helped me overcome setbacks and stay focused on my graduate study. I greatly value their friendship and I deeply appreciate their belief in me.

Most importantly, none of this would have been possible without the love and endless care of my family: my father, my mother, and my brother. I would like to thank my parents for giving birth to me at the first place and supporting me spiritually throughout my life.

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CHAPTER 1 INTRODUCTION

The dissertation is devoted to the study and applications of a new class of optimal control problems governed by a perturbed sweeping process of the hysteresis type with control functions acting in both play-and-stop operator and additive perturbations. Such control problems can be reduced to optimization of discontinuous and unbounded differential inclusions with pointwise state constraints, which are immensely challenging in control theory and prevent employing conventional variation techniques to derive necessary optimality conditions. We develop the method of discrete approximations married with appropriate generalized differential tools of modern variational analysis to overcome principal difficulties in passing to the limit from optimality conditions for finite-difference systems. This approach leads us to nondegenerate necessary conditions for local minimizers of the controlled sweeping process expressed entirely via the problem data. Besides illustrative examples, we apply the obtained results to an optimal control problem associated with of the crowd motion model of traffic flow in a corridor, which is formulated in this thesis. The derived optimality conditions allow us to develop an effective procedure to solve this problem in a general setting and completely calculate optimal solutions in particular situations.

In this work, we deal with a version of the *sweeping process* introduced by Jean-Jacques Moreau in the 1970s (see his comprehensive for that time lecture notes [40] with the references to the original publications) in the following form of the dissipative differential inclusion:

$$-\dot{x}(t) \in N\big(x(t); C(t)\big) \text{ a.e. } t \in [0,T], \quad x(0) := x_0 \in C(0) \subset H, \tag{1.1}$$

where $C(t)$ is a continuously moving convex set, and where the normal cone operator to a convex subset $C \subset H$ of a Hilbert space is given by

$$N(x;C) := \big\{v \in H \big|\ \langle v, y-x\rangle \le 0,\ y \in C\big\} \text{ if } x \in C \text{ and } N(x;C) := \emptyset \text{ if } x \notin C. \quad (1.2)$$

The latter construction allows us to equivalently describe (1.1) as an *evolution variational inequality* [7,24], or as a *differential variational inequality* in the terminology of [43,48]. The original motivations for the introduction and study of the sweeping process come from applications to mechanical systems mostly related to friction and elastoplasticity, while further developments apply also to various problems of hysteresis, ferromagnetism, electric circuits, phase transitions, economics, etc.; see, e.g., [2,7,23,24,39,47,48] and the extensive bibliographies therein.

Our work in this thesis concerns a new class of optimal control problems for the *perturbed sweeping process* (see [9,10])

$$-\dot{x}(t) \in N\big(x(t);C(t)\big) + f\big(x(t),a(t)\big) \text{ a.e. } t \in [0,T], \quad x(0) := x_0 \in C(0), \quad (1.3)$$

with one part of controls $a\colon [0,T] \to \mathbb{R}^d$ acting in the perturbation mapping $f\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and the other part of controls $u\colon [0,T] \to \mathbb{R}^n$ acting in the moving set

$$C(t) := C + u(t) \text{ with } C := \big\{x \in \mathbb{R}^n \big|\ \langle x_i^*, x\rangle \le 0 \text{ for all } i = 1,\ldots,m\big\}, \quad (1.4)$$

where x_i^* are fixed vectors from $\mathbb{R}^n$, and where the final time $T > 0$ is also fixed. The minimizing cost functional is given in the generalized Bolza form

$$\text{minimize } J[x,u,a] := \varphi\big(x(T)\big) + \int_0^T \ell\big(t, x(t), u(t), a(t), \dot{x}(t), \dot{u}(t), \dot{a}(t)\big)\, dt \quad (1.5)$$

with the proper terminal extended-real-valued cost function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ and the running cost function $\ell : [0,T] \times \mathbb{R}^{4n+2d} \to \overline{\mathbb{R}}$. Fixed $r > 0$, we impose the additional constraint on the u-controls:

$$\|u(t)\| = r \quad \text{for all} \quad t \in [0, T] \tag{1.6}$$

required by applications. The primary application we have in mind is the *crowd motion model* (see, e.g., [30]), which corresponds to (1.3) with controls only in perturbations and whose simplified optimal control version is solved in our paper [10] based on the obtained optimality conditions.

Besides the dynamic constraints (1.3), problem (P^τ) involves the pointwise constraints on u-controls:

$$\begin{cases} \|u(t)\| = r \quad \text{for all} \quad t \in [\tau, T - \tau], \\ r - \tau \leq \|u(t)\| \leq r + \tau \quad \text{for all} \quad t \in [0, \tau) \cup (T - \tau, T] \end{cases} \tag{1.7}$$

depending on the parameter $\tau \in [0, \overline{\tau}]$ with $\overline{\tau} := \min\{r, T\}$ and fixed $r > 0$. Note that the inclusion in (1.3) and the second part of definition (1.2) implicitly yield the pointwise constraints of another type

$$\big\langle x_i^*, x(t) - u(t) \big\rangle \leq 0 \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad i = 1, \ldots, m. \tag{1.8}$$

The characteristic feature of problem (P^τ) for any fixed $\tau \in [0, \overline{\tau}]$ is the differential inclusion (1.3) describing, for each fixed control pair $(u(\cdot), a(\cdot))$, a perturbed version of Moreau's *sweeping process* [40] the mathematical theory of which has been well developed; see, e.g., [11, 20, 25] and the references therein. The sweeping inclusion (1.3) significantly differs from those considered in optimal control theory for differential inclusions as developed in [4, 18, 37, 52] and other publications, since (1.3) admits a unique solution $x(\cdot)$ whenever the sweeping set $C(\cdot)$ and the perturbation function $a(\cdot)$ therein are given a priori; and so there is no room for optimization in such a case. Our control model in (P^τ) follows the line of [12, 13], where control actions enter the sweeping set but not entering perturbations. Other optimal control problems for various versions of the sweeping process are considered in [1,8,20] with no controls in the sweeping set. Namely, [20] deals with controls only in perturbations addressing existence and relaxation issues for optimal solutions, while [1,8] apply controls in associated differential equations with deriving necessary optimality conditions for discrete-time [1] and continuous-time [8] systems.

The *main goal* of our work [9] is to study the formulated optimal control problems (P) and (P^τ) in what follows by using the *method of discrete approximations* in the vein of [35,37] and its significant modification for the case of unperturbed non-Lipschitzian differential inclusions developed in [13]. The presence of controlled perturbations in (1.3) together with the mixed constraints $\langle x_i^*, x(t) - u(t)\rangle \le 0$ essentially complicates the discrete approximation procedure. We constructed well-posed discrete approximations in such a way that every *feasible* (resp. *locally optimal*) solution to (P^τ) with $\tau \ge 0$ and $(P^0) = (P)$, can be *strongly approximated* in $W^{1,2}[0,T]$ by feasible (resp. optimal) solutions to finite-difference control systems. Employing then appropriate first-order and second-order generalized differential constructions of variational analysis and explicitly calculating them via the problem data allow us to successfully obtain effective *necessary optimality conditions* for discrete optimal solutions, which can be treated as *suboptimality* (almost optimality) conditions for the original sweeping control problem.

The paper [10] can be considered a continuation of our work in [9]. The major goal of this paper is to derive nondegenerate necessary optimality conditions for the so-called intermediate (including strong) local minimizers of the sweeping control problems under consideration by passing to the limit from the necessary optimality conditions for their discrete approximations obtained in [9].

In contrast to all the previous developments, we address in this thesis *necessary optimality conditions* for problem (P^τ) with controls in *both sweeping set* and *additive perturbations*. Note that the structure of the sweeping set in (1.3), (1.4) is specific for the so-called *play-and-stop operator* [47] and largely relates to *rate independent hysteresis*; see, e.g., [24,32,47]. Our main application here is given to a corridor version of the *crowd motion model* [30,51], where introducing controls in perturbations allows us to optimize the corresponding sweeping process and determine the optimal strategy of crowd motion participants.

Considering the triple $z = (x, u, a) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$, it is easy to observe that (1.3) can be written as

$$-\dot{z}(t) \in F\big(z(t)\big) \times \mathbb{R}^n \times \mathbb{R}^d \;\text{ a.e. }\; t \in [0,T] \;\text{ with }\; F(z) := N(x-u;C) + f(x,a), \quad (1.9)$$

where the initial triple $z(0) = (x_0, u(0), a(0))$ satisfies the condition $x_0 - u(0) \in C$ via the convex polyhedron C defined in (1.4). Then the sweeping optimal control problem (P^τ) amounts to minimizing the cost functional $J[z] = J[x, u, a]$ in (1.5) over $W^{1,2}$-solutions to the differential inclusion (1.5) subject to the pointwise *state constraints* of the equality and inequality types in (1.7) and (1.8), where the latter ones are implicit from (1.5). Although problem (P^τ) is now written in the usual form of the theory of differential inclusions, it is far removed from satisfying the assumptions under which necessary optimality conditions have been developed in this theory. First of all, the right-hand side of (1.5) is intrinsically *unbounded*, *discontinuous*, and *highly non-Lipschitzian* in any generalized sense treated by the developed approaches of optimal control theory for differential inclusions. Furthermore, besides the inequality state constraints in (1.8), problem (P^τ) contains the unconventional *equality* ones as in (1.7). Such constraints have just recently started to be considered in control theory for smooth ordinary differential equations [5], where necessary optimality conditions are obtained under strong regularity assumptions including full rank of the smooth constraint Jacobians, which is not the case in (1.7).

In this thesis, we develop the *method of discrete approximations* to derive necessary optimality conditions for control problems governed by differential inclusions following the scheme of [35, 36], where the discrete approximation approach is realized for Lipschitzian differential inclusions without state constraints, and then its recent significant modification given in [13] in the case of the sweeping process with general polyhedral controlled sets but without control actions in additive perturbations. Note that our developments in this work result in new optimality conditions that have important advantages in comparison with those in [13] even in the case of no controls in perturbations.

CHAPTER 2 OPTIMAL CONTROL OF A PERTURBED SWEEPING PROCESS VIA DISCRETE APPROXIMATIONS

In this chapter, we develop the method of discrete approximations, which allows us to adequately replace the original optimal control problem (P^τ) by a sequence of well-posed finite-dimensional optimization problems whose optimal solutions strongly converge to that of the controlled perturbed sweeping process. To solve the discretized control systems, we derive effective necessary optimality conditions by using second-order generalized differential tools of variational analysis that explicitly calculated in terms of the given problem data.

2.1 Standing Assumptions and Preliminaries

Throughout the work we impose the following standing assumptions on the initial data of the optimal control problem (P) in (1.3)–(1.7):

(H1) The mapping $f\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n \times \mathbb{R}^d$ and locally Lipschitz continuous in the first argument, i.e., for every $\varepsilon > 0$ there is a constant $K > 0$ such that

$$\|f(x,a) - f(y,a)\| \le K\|x-y\| \text{ whenever } (x,y) \in B(0,\varepsilon) \times B(0,\varepsilon), \quad a \in \mathbb{R}^d. \tag{2.1}$$

Furthermore, there is a constant $M > 0$ ensuring the growth condition

$$\|f(x,a\| \le M\big(1 + \|x\|\big) \text{ for any } x \in \bigcup_{t\in[0,T]} C(t), \quad a \in \mathbb{R}^d. \tag{2.2}$$

(H2) The terminal cost function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ and the running cost function $\ell : [0,T] \times \mathbb{R}^{4n+2d} \to \overline{\mathbb{R}}$ in (1.5) are lower semicontinuous (l.s.c.) while ℓ is bounded from below on bounded sets.

Now we are ready to formulate the powerful well-posedness result for the sweeping process under consideration that reduces to [20, Theorem 1].

Proposition 2.1 (well-posedness of the controlled sweeping process). *Under the assumptions in* (H1)*, let* $u(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ *and* $a(\cdot) \in W^{1,2}([0,T];\mathbb{R}^d)$*, and let* $M > 0$ *be taken from* (2.2)*. Then the perturbed sweeping inclusion* (1.6) *with* $C(t)$ *from* (1.7) *admits the unique solution* $x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ *generated by* $(u(\cdot),a(\cdot))$ *and satisfying the estimates*

$$\|x(t)\| \le l := \|x_0\| + e^{2MT}\Big(2MT(1+\|x_0\|) + \int_0^T \|\dot u(s)\|ds\Big) \;\; \textit{for all} \;\; t \in [0,T], \qquad (2.3)$$

$$\|\dot x(t)\| \le 2(1+l)M + \|\dot u(t)\| \;\; \textit{a.e.} \;\; t \in [0,T].$$

Proof. To deduce this result from [20, Theorem 1], with taking into account the solution estimates therein, it remains to verify that $C(t)$ in (1.7) generated by the chosen $W^{1,2}$-control $u(\cdot)$ varies in an *absolutely continuous way* [20], i.e., there is an absolutely continuous function $v\colon [0,T] \to \mathbb{R}$ such that

$$\big|\mathrm{dist}\big(y;C(t)\big) - \mathrm{dist}\big(y;C(s)\big)\big| \le |v(t)-v(s)| \;\; \text{for all} \;\; t,s \in [0,T] \qquad (2.4)$$

with $\mathrm{dist}(x;\Omega)$ standing for the distance from $x \in \mathbb{R}^n$ to the closed set $\Omega \subset \mathbb{R}^n$ and with the function

$$v(t) := \int_0^t \|\dot u(s)\|ds, \quad 0 \le t \le T,$$

in our case. To verify (2.4), pick any $y \in \mathbb{R}^n$ and $c \in C$ and then easily get the estimates

$$\mathrm{dist}\big(y;C(t)\big) = \mathrm{dist}\big(y;u(t)+C\big) \le \|y-u(t)-c\| \le \|y-u(s)-c\| + \|u(t)-u(s)\|,$$

which imply in turn by the definition of the distance function that

$$\mathrm{dist}\big(y;C(t)\big) \le \inf_{c\in C}\|y-u(s)-c\| + \|u(t)-u(s)\| = \mathrm{dist}\big(y;C(s)\big) + \|u(t)-u(s)\|.$$

Using this and then changing the positions of t and s give us the resulting inequality

$$\big|\text{dist}\big(y;C(t)\big)-\text{dist}\big(y;C(s)\big)\big|\le\|u(t)-u(s)\|\quad\text{for all}\ \ t,s\in[0,T].$$

This finally yields (2.4) by observing that

$$\begin{aligned}\big|d\big(y;C(t)\big)-d\big(y;C(s)\big)\big|&\le\|u(t)-u(s)\|=\left\|\int_s^t\dot u(\theta)d\theta\right\|\\&\le\int_s^t\|\dot u(\theta)\|d\theta=\int_s^t\dot v(\theta)d\theta=|v(t)-v(s)|\end{aligned}$$

and thus completes the proof of the proposition. □

2.2 Discrete Approximations of Feasible Solutions

In this section we construct a sequence of discrete approximations of the sweeping differential inclusion in (1.6), (1.7) with the constraints in (1.9) and (1.8), but without appealing to the minimizing functional (1.5). The main result of this section is justifying the strong $W^{1,2}$-approximation of *any* feasible control and the corresponding sweeping trajectory by their finite-difference counterparts, which are piecewise linearly extended to the continuous-time interval $[0,T]$.

First we reduce (1.6) to a more conventional form of differential inclusions. Introduce the new variable $z:=(x,u,a)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d$ and define the set-valued mapping $F:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d\rightrightarrows\mathbb{R}^n$ by

$$F(z)=F(x,u,a):=N(x-u;C)+f(x,a).\tag{2.5}$$

Consider the collection of *active constraint indices* of polyhedron (1.7) at $\bar x\in C$ given by

$$I(\bar x):=\big\{i\in\{1,\dots,m\}\big|\ \big\langle x_i^*,\bar x\big\rangle=0\big\},\tag{2.6}$$

it is not difficult to observe (see, e.g., [22, Proposition 3.1]) the explicit representation

$$F(z) = \Big\{ \sum_{i \in I(x-u)} \lambda_i x_i^* \Big| \ \lambda_i \geq 0 \Big\} + f(x,a) \tag{2.7}$$

of (2.5) via the active index set (2.6) at $x - u \in C$. Then we can rewrite (1.6) in the following equivalent form with respect to the variable $z = (x, u, a)$:

$$-\dot{z}(t) \in F\big(z(t)\big) \times \mathbb{R}^n \times \mathbb{R}^d \ \text{ a.e. } \ t \in [0, T] \tag{2.8}$$

with the initial condition $z(0) = (x_0, u(0), a(0))$ satisfying $x_0 - u(0) \in C$, i.e., such that $\langle x_i^*, x_0 - u(0) \rangle \leq 0$ for all $i = 1, \ldots, m$. Proposition 2.1 allows us to have solutions of the differential inclusion (2.8) in the class of $W^{1,2}$-functions $z(t) = (x(t), u(t), a(t))$ on $[0, T]$.

Note that, although the resulting system (2.8) is written in the conventional form of the theory of differential inclusions, it does not satisfy usual assumptions therein. Indeed, the right-hand side of (2.8) is intrinsically *unbounded* in all its components, including the first (perturbed normal cone) one in which is *highly non-Lipschitzian*. Furthermore, the constrained system under consideration contains the intrinsic inequality *state constraints* (1.8) together with the equality one (1.9) on the whole time interval $[0, T]$.

Having in mind further applications including those developed in [10], it makes sense to consider a *parametric version* of the equality constraint in (1.9) with a small parameter $\tau \geq 0$ while replacing (1.9) by

$$\begin{cases} \|u(t)\| = r \ \text{ for all } \ t \in [\tau, T - \tau], \\ r - \tau \leq \|u(t)\| \leq r + \tau \ \text{ for all } \ t \in [0, \tau) \cup (T - \tau, T], \end{cases} \tag{2.9}$$

which reduces to (1.9) when $\tau = 0$. Fix any $\tau \in [0, \min\{r, T\}]$, $k \in I\!N$ and denote by $j_\tau(k) := [k\tau/T]$ the smallest index j such that $t_j^k \geq \tau$ and by $j^\tau(k) := [k(T - \tau)/T] - 1$ the largest j with $t_j^k \le T - \tau$.

The next theorem on the *strong discrete approximation* of feasible sweeping solutions is a counterpart of [13, Theorem 3.1] for the perturbed sweeping process in (1.6), (1.7) constrained by (1.8), (2.9) with additional *quantitative estimates* expressed via the system data. The reader can see that both the formulation and proof in the new setting are significantly more involved in comparison with [13]. Observe also the *novel approximation conclusion* (2.13), which holds also in the setting of [13] while being missed therein. This conclusion will allow us to construct a more precise discrete approximation of a local minimizer in Theorem 2.5, which is crucial to derive a new transversality condition for the original continuous-time sweeping control problem (P^τ) in [10].

Theorem 2.2 **($W^{1,2}$-strong discrete approximation of feasible sweeping solutions).** *Under the validity of* (H1)*, let the triple* $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot))$ *be a given feasible solution to the constrained sweeping system from* (1.6)*,* (1.7)*, and* (2.9) *with a fixed parameter* $\tau \in [0, \min\{r, T\}]$*, and let the constant* K *be taken from* (2.1)*. Define the discrete partitions of* $[0, T]$ *by*

$$\Delta_k := \left\{0 = t_0^k < t_1^k < \ldots < t_k^k\right\} \;\; \textit{with} \;\; h_k := t_{j+1}^k - t_j^k \downarrow 0 \;\; \textit{as} \;\; k \to \infty \tag{2.10}$$

and suppose that $\bar{z}(\cdot)$ *has the following properties at the mesh points* (*while observing that all these properties hold if* $\bar{z}(\cdot) \in W^{2,\infty}[0, T]$)*: it satisfies* (2.8) *at* t_j^k *as* $j = 0, \ldots, k-1$ *for all* $k \in I\!N$ (*with the right-side derivative at* $t_0 = 0$)*, we have*

$$\sum_{j=0}^{k-1} (t_{j+1}^k - t_j^k) \left\| \frac{\bar{x}(t_{j+1}^k) - \bar{x}(t_j^k)}{t_{j+1}^k - t_j^k} - \dot{\bar{x}}(t_j^k) \right\|^2 \to 0 \;\; \textit{as} \;\; k \to \infty, \tag{2.11}$$

and there is a constant $\mu > 0$ independent of k such that

$$\sum_{j=0}^{k-1}\left\|\frac{\bar{x}(t_{j+1}^k)-\bar{x}(t_j^k)}{h_k}-\dot{\bar{x}}(t_j^k)\right\|\le\mu,\quad\left\|\frac{\bar{u}(t_1^k)-\bar{u}(t_0^k)}{h_k}\right\|\le\mu,$$
$$\sum_{j=0}^{k-2}\left\|\frac{\bar{u}(t_{j+2}^k)-\bar{u}(t_{j+1}^k)}{h_k}-\frac{\bar{u}(t_{j+1}^k)-\bar{u}(t_j^k)}{h_k}\right\|\le\mu. \tag{2.12}$$

Then there exist a sequence of piecewise linear functions $z^k(t) := (x^k(t), u^k(t), a^k(t))$ on $[0,T]$ and a sequence of $\varepsilon_k \le 2h_k\mu e^K \downarrow 0$ as $k\to\infty$ for which $(x^k(0), u^k(0), a^k(0)) = (x_0, \bar{u}(0), \bar{a}(0))$,

$$\frac{x^k(t_1^k)-x^k(t_0^k)}{h_k}\to\dot{\bar{x}}(0)\quad as\quad k\to\infty, \tag{2.13}$$

$$\begin{cases}\|u^k(t_j^k)\|=r\quad if\;\; j=j_\tau(k),\ldots,j^\tau(k),\\ r-\tau-\varepsilon_k\le\|u^k(t_j^k)\|\le r+\tau+\varepsilon_k\quad if\;\; j=0,\ldots,j_\tau(k)-1\;\; and\;\; j\ge j^\tau(k)+1,\end{cases} \tag{2.14}$$

$$x^k(t)=x^k(t_j)-(t-t_j)v_j^k,\; x^k(0)=x_0,\; t_j^k\le t\le t_{j+1}^k\;\; with\;\; v_j^k\in F\big(z^k(t_j^k)\big),\quad j=0,\ldots,k-1, \tag{2.15}$$

and the functions $z^k(\cdot)$ converge to $\bar{z}(\cdot)$ in the norm topology of $W^{1,2}[0,T]$, i.e.,

$$z^k(t)\to\bar{z}(t)\;\; uniformly\ on\;\; [0,T]\;\; and\;\; \int_0^T\|\dot{z}^k(t)-\dot{\bar{z}}(t)\|^2dt\to 0\;\; as\;\; k\to\infty. \tag{2.16}$$

Furthermore, for every $k\in I\!N$ we have the estimates

$$\mathrm{var}\,\big(\dot{u}^k;[0,T]\big)\le\widetilde{\mu}\;\; and\;\; \left\|\frac{u^k(t_1^k)-u^k(t_0^k)}{h_k}\right\|\le\widetilde{\mu}\;\; with\;\; \widetilde{\mu}:=\max\big\{3\mu(1+4KT)e^K,4\mu(e^K+1)\big\}, \tag{2.17}$$

where the symbol "var" stands for the total variation of the function in question.

Proof. Let $y^k(\cdot) := (y_1^k(\cdot), y_2^k(\cdot), y_3^k(\cdot))$ be piecewise linear on $[0,T]$ and such that

$$\big(y_1^k(t_j^k),y_2^k(t_j^k),y_3^k(t_j^k)\big):=\big(\bar{x}(t_j^k),\bar{u}(t_j^k),\bar{a}(t_j^k)\big)\,,\quad j=0,\ldots,k. \tag{2.18}$$

Define next $w^k(t) = (w_1^k(t), w_2^k(t), w_3^k(t)) := \dot{y}^k(t)$ as piecewise constant and right continuous function on $[0,T]$ via the derivatives at non-mesh points and deduce from (2.12) that $\text{var}\,(w_2^k; [0,T]) \le \mu$ for every $k \in I\!N$. It follows from the definition of $w^k(\cdot)$ that

$$w_1^k(0) = \frac{\bar{x}(t_1^k) - \bar{x}(t_0^k)}{h_k} \to \dot{\bar{x}}(0) \ \text{ as } \ k \to \infty \tag{2.19}$$

due to the existence of the right derivative of $\dot{\bar{x}}(0)$ by the imposed assumption on the validity of (2.8) at the mesh points. Furthermore, we get from (1.8) that

$$\langle x_i^*, y_1^k(t_j^k) - y_2^k(t_j^k)\rangle = \langle x_i^*, \bar{x}(t_j^k) - \bar{u}(t_j^k)\rangle \le 0$$

on the mesh Δ_k for all $j = 1, \ldots, k-1$ and $i = 1, \ldots, m$. The constructions made ensure that

$$y^k(\cdot) \to \bar{z}(\cdot) \ \text{ uniformly on } \ [0,T] \ \text{ and } \ w^k(\cdot) \to \dot{\bar{z}}(\cdot) \ \text{ in norm of } \ L^2([0,T]; \mathbb{R}^{2n+d}).$$

Denote $a^k(t) := y_3^k(t)$ for all $t \in [0,T]$, fix $k \in I\!N$, and use for simplicity the notation $t_j := t_j^k$ as $j = 1, \ldots, k$. To construct the claimed trajectories $x^k(t)$ of (2.15), we proceed by induction and suppose that the value of $x^k(t_j)$ is known. Define now the vectors

$$u^k(t_j) := x^k(t_j) - y_1^k(t_j) + y_2^k(t_j) = x^k(t_j) - \bar{x}(t_j) + \bar{u}(t_j), \quad j = 0, \ldots, k,$$

and assume without loss of generality that $\|u^k(t_j)\| = r$ for $j = j_\tau(k), \ldots, j^\tau(k)$, which clearly yields

$$x^k(t_j) - u^k(t_j) = \bar{x}(t_j) - \bar{u}(t_j) \ \text{ for } \ j = 0, \ldots, k. \tag{2.20}$$

Since the sets $F(z)$ in (2.5) are closed and convex, we select the unique projection

$$v_j^k := \Pi\big(-w_{1j}^k; F\big(x^k(t_j), u^k(t_j), a^k(t_j)\big)\big), \quad j = 0, \ldots, k, \tag{2.21}$$

and deduce from (2.19) that $v_0^k \to \dot{\bar{x}}(0)$ as $k \to \infty$. Defining next $x^k(t) := x^k(t_j) - (t-t_j)v_j^k$ for all $t \in [t_j, t_{j+1}]$ and $j = 0, \ldots, k$ shows that the inclusions in (2.15) are fulfilled and condition (2.13) holds. Furthermore, we deduce from (2.7) and (2.20) that

$$F\big(x^k(t_j), u^k(t_j), a^k(t_j)\big) = F\big(\bar{x}(t_j), \bar{u}(t_j), \bar{a}(t_j)\big) + f\big(x^k(t_j), \bar{a}(t_j)\big) - f\big(\bar{x}(t_j), \bar{a}(t_j)\big) \quad (2.22)$$

at the mesh points. To verify that the triples $(x^k(t), u^k(t), a^k(t))$, $k \in I\!N$, constructed above satisfy all the conclusions of the theorem, let us first show that

$$\varepsilon_k := \big\|x^k(t_j) - \bar{x}(t_j)\big\| \le \max\big\{h_k\mu(1+h_kK), 2h_k\mu e^K\big\} = 2h_k\mu e^K \quad (2.23)$$

for all $j = 0, \ldots, k$. Indeed, picking any $t \in [t_j, t_{j+1}]$ for $j = 0, \ldots, k-1$, we have the representation

$$x^k(t) - y_1^k(t) = x^k(t_j) - \bar{x}(t_j) + (t-t_j)(-v_j^k - w_{1j}^k),$$

which implies in turn the estimate

$$\begin{aligned}\big\|x^k(t) - y_1^k(t)\big\| &\le \big\|x^k(t_j) - \bar{x}(t_j)\big\| + (t_{j+1}-t_j)\,\big\|-v_j^k - w_{1j}^k\big\| \\ &= \big\|x^k(t_j) - \bar{x}(t_j)\big\| + (t_{j+1}-t_j)\mathrm{dist}\left(-\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1}-t_j}; F\big(z^k(t_j)\big)\right).\end{aligned}$$

It then follows from (2.22) that

$$\begin{aligned}\big\|x^k(t) - y_1^k(t)\big\| \le \big\|x^k(t_j) - \bar{x}(t_j)\big\| &+ h_k\,\big\|f\big(x^k(t_j), \bar{a}(t_j)\big) - f\big(\bar{x}(t_j), \bar{a}(t_j)\big)\big\| \\ &+ h_k\left\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1}-t_j} - \dot{\bar{x}}(t_j)\right\|.\end{aligned}$$

Using the Lipschitz continuity of f with respect to x imposed in (2.1) gives us

$$\big\|x^k(t) - y_1^k(t)\big\| \le (1+h_kK)\,\big\|x^k(t_j) - \bar{x}(t_j)\big\| + h_k\left\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1}-t_j} - \dot{\bar{x}}(t_j)\right\|, \quad (2.24)$$

and thus, by taking the first condition in (2.12) into account, we arrive at the inequalities

$$
\begin{cases}
\left\|x^k(t_1)-\bar{x}(t_1)\right\| \le h_k\left\|\dfrac{\bar{x}(t_1)-\bar{x}(t_0)}{t_1-t_0}-\dot{\bar{x}}(t_0)\right\|, \\
\left\|x^k(t_2)-\bar{x}(t_2)\right\| \le (1+h_kK)\left\|x^k(t_1)-\bar{x}(t_1)\right\| + h_k\left\|\dfrac{\bar{x}(t_2)-\bar{x}(t_1)}{t_2-t_1}-\dot{\bar{x}}(t_1)\right\| \\
\le h_k\left(\left\|\dfrac{\bar{x}(t_1)-x(t_0)}{t_1-t_0}-\dot{x}(t_0)\right\| + \left\|\dfrac{\bar{x}(t_2)-\bar{x}(t_1)}{t_2-t_1}-\dot{\bar{x}}(t_1)\right\|\right) + h_k^2K\mu.
\end{cases}
\tag{2.25}
$$

Now we proceed by induction to verify that

$$
\begin{aligned}
\left\|x^k(t_j)-\bar{x}(t_j)\right\| \le{}& h_k\sum_{i=0}^{j-1}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\| \\
&+ h_k^2K\mu\sum_{i=0}^{j-3}(1+h_kK)^i + (1+h_kK)^{j-1}h_k\mu
\end{aligned}
\tag{2.26}
$$

for $j = 3, \ldots, k$. Starting with $j = 3$, observe from (2.12), (2.24), and (2.25) that

$$
\begin{aligned}
\left\|x^k(t_3)-\bar{x}(t_3)\right\| \le{}& (1+h_kK)\left\|x^k(t_2)-\bar{x}(t_2)\right\| + h_k\left\|\frac{\bar{x}(t_3)-\bar{x}(t_2)}{t_3-t_2}-\dot{\bar{x}}(t_2)\right\| \\
\le{}& (1+h_kK)\left(h_k\sum_{i=0}^{1}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\| + h_k^2K\mu\right) \\
&+ h_k\left\|\frac{\bar{x}(t_3)-\bar{x}(t_2)}{t_3-t_2}-\dot{\bar{x}}(t_2)\right\| \\
={}& h_k\sum_{i=0}^{2}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\| + h_k^2K\sum_{i=0}^{1}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\| \\
&+ (1+h_kK)h_k^2K\mu \\
\le{}& h_k\sum_{i=0}^{2}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\| + h_k^2K\mu + (1+h_kK)^2h_k\mu,
\end{aligned}
$$

which justifies the validity of (2.26) at $j = 3$. Suppose next that (2.26) holds for t_j as $j \ge 3$

and show that it is also satisfied for t_{j+1}. Indeed, employing (2.12) and (2.24) tells us that

$$\begin{aligned}\left\|x^k(t_{j+1})-\bar{x}(t_{j+1})\right\| =& \left\|x^k(t_{j+1})-y_1^k(t_{j+1})\right\|\\ \le&(1+h_kK)\left\|x^k(t_j)-\bar{x}(t_j)\right\|+h_k\left\|\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{t_{j+1}-t_j}-\dot{\bar{x}}(t_j)\right\|\\ \le&(1+h_kK)\left(h_k\sum_{i=0}^{j-1}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\|+h_k^2K\mu\sum_{i=0}^{j-3}(1+h_kK)^i\right)\\ &+(1+h_kK)^jh_k\mu+h_k\left\|\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{t_{j+1}-t_j}-\dot{\bar{x}}(t_j)\right\|\\ \le&h_k\sum_{i=0}^{j}\left\|\frac{\bar{x}(t_{i+1})-\bar{x}(t_i)}{t_{i+1}-t_i}-\dot{\bar{x}}(t_i)\right\|+h_k^2K\mu\sum_{i=0}^{j-2}(1+h_kK)^i+(1+h_kK)^jh_k\mu,\end{aligned}$$

which shows that estimate (2.26) holds for t_{j+1}, and thus it is justified for all $j=3,\ldots,k$.

Now picking any $j\in\{3,\ldots,k\}$ and using the first inequality in (2.12), we get

$$\begin{aligned}\left\|x^k(t_j)-\bar{x}(t_j)\right\| \le&h_k\mu+h_k\mu[(1+h_kK)^{j-2}-1]+(1+h_kK)^kh_k\mu\\ \le&2h_k\mu(1+h_kK)^k\\ =&2h_k\mu\left(1+\frac{K}{k}\right)^k\\ \le&2h_k\mu e^K.\end{aligned}$$

Combining it with (2.25), we arrive at (2.23). This readily implies that

$$r-\tau-\varepsilon_k\le\|u^k(t_j)\|\le r+\tau+\varepsilon_k$$

for $j\le j_\tau(k)-1$ and $j\ge j^\tau(k)+1$, i.e., the relationships in (2.14) are satisfied with ε_k defined in (2.23). Furthermore, it follows from (2.23) and (2.24) that

$$\begin{aligned}\left\|x^k(t)-y_1^k(t)\right\| \le&(1+h_kK)2h_k\mu e^K+h_k\left\|\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{t_{j+1}-t_j}-\dot{\bar{x}}(t_i)\right\|\\ \le&2h_k\mu e^K(1+h_kK)+h_k\mu\end{aligned} \tag{2.27}$$

for $t\in[t_j,t_{j+1}]$ and $j=0,\ldots,k-1$. Next we consider relationships for the u-component of

$z^k(\cdot)$. The first and third conditions in (2.12) yield

$$\begin{aligned}
&\sum_{j=0}^{k-2}\left\|\frac{u^k(t_{j+2})-u^k(t_{j+1})}{t_{j+2}-t_{j+1}}-\frac{u^k(t_{j+1})-u^k(t_j)}{t_{j+1}-t_j}\right\| \\
&\quad\le\sum_{j=0}^{k-2}\left\|\frac{\bar u(t_{j+2})-\bar u(t_{j+1})}{t_{j+2}-t_{j+1}}-\frac{\bar u(t_{j+1})-\bar u(t_j)}{t_{j+1}-t_j}\right\| \\
&\quad+2\sum_{j=0}^{k-1}\left\|\frac{x^k(t_{j+1})-x^k(t_j)}{t_{j+1}-t_j}-\frac{\bar x(t_{j+1})-\bar x(t_j)}{t_{j+1}-t_j}\right\| \\
&\quad\le\mu+2\sum_{j=0}^{k-1}\left\|-v_j^k-\frac{\bar x(t_{j+1})-\bar x(t_j)}{t_{j+1}-t_j}\right\| \\
&\quad=\mu+2\sum_{j=0}^{k-1}\left\|-v_j^k-w_{1j}^k\right\| \\
&\quad=\mu+2\sum_{j=0}^{k-1}\operatorname{dist}\left(-\frac{\bar x(t_{j+1})-\bar x(t_j)}{t_{j+1}-t_j};F\big(z^k(t_j)\big)\right) \\
&\quad\le\mu+2\sum_{j=0}^{k-1}\left\|\frac{\bar x(t_{j+1})-\bar x(t_j)}{t_{j+1}-t_j}-\dot{\bar x}(t_j)\right\|+2\sum_{j=0}^{k-1}K\left\|x^k(t_j)-\bar x(t_j)\right\| \\
&\quad\le3\mu+2Kk2h_k\mu e^K \\
&\quad=3\mu+4KT\mu e^K \\
&\quad\le\widetilde\mu=\max\left\{3\mu+4KT\mu e^K,4\mu e^K+\mu\right\},
\end{aligned}$$

which justifies the first estimate in (2.17). To verify the second estimate therein, we deduce from (2.20), (2.23), and the second inequality in (2.12) that

$$\begin{aligned}
\left\|\frac{u^k(t_1)-u^k(t_0)}{t_1-t_0}\right\|&\le\left\|\frac{u^k(t_1)-\bar u(t_1)}{t_1-t_0}\right\|+\left\|\frac{u^k(t_0)-\bar u(t_0)}{t_1-t_0}\right\|+\left\|\frac{\bar u(t_1)-\bar u(t_0)}{t_1-t_0}\right\| \\
&\le\left\|\frac{x^k(t_1)-\bar x(t_1)}{t_1-t_0}\right\|+\left\|\frac{x^k(t_0)-\bar x(t_0)}{t_1-t_0}\right\|+\mu \\
&\le4\mu e^K+\mu \\
&\le\widetilde\mu,
\end{aligned}$$

which readily gives us the claimed result in (2.17).

It remains to justify the $W^{1,2}$-convergence of $z^k(t)$ to $\bar{z}(t)$ in (2.16). Using (2.20) for $j = 0$ with $x^k(t_0) = x_0$, the construction of $a^k(t)$, and the Newton-Leibniz formula, it suffices to show that the sequence of $(\dot{x}^k(t), \dot{u}^k(t))$ converges to $(\dot{\bar{x}}(t), \dot{\bar{u}}(t))$ strongly in $L^2[0,T]$. To this end we have

$$\begin{aligned}
\int_0^T \left\|\dot{x}^k(t) - w_1^k(t)\right\|^2 dt =& \sum_{j=0}^{k-1} (t_{j+1} - t_j) \left\|-v_j^k - w_{1j}^k\right\|^2 \\
=& \sum_{j=0}^{k-1} h_k \text{dist}^2 \left(-\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j}; F\big(z^k(t_j)\big)\right) \\
\le& \sum_{j=0}^{k-1} h_k \left(K \left\|x^k(t_j) - \bar{x}(t_j)\right\| + \left\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j} - \dot{\bar{x}}(t_j)\right\|\right)^2 \\
\le& 2\sum_{j=0}^{k-1} h_k K^2 \left\|x^k(t_j) - \bar{x}(t_j)\right\|^2 + 2\sum_{j=0}^{k-1} h_k \left\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j} - \dot{\bar{x}}(t_j)\right\|^2 \\
\le& 2\sum_{j=0}^{k-1} h_k K^2 \left(2h_k \mu e^K\right)^2 + 2\sum_{j=0}^{k-1} h_k \left\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j} - \dot{\bar{x}}(t_j)\right\|^2 \\
\le& 8TK^2 h_k^2 \mu^2 e^{2K} + 2\sum_{j=0}^{k-1} h_k \left\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j} - \dot{\bar{x}}(t_j)\right\|^2 \to 0
\end{aligned}$$

as $k \to \infty$ due to (2.11), (2.18), and the definition of $w^k(t)$. It follows furthermore that

$$\begin{aligned}
\int_0^T \left\|\dot{u}^k(t) - w_2^k(t)\right\|^2 dt =& \int_0^T \left\|\frac{u^k(t_{j+1}) - u^k(t_j)}{h_k} - \frac{\bar{u}(t_{j+1}) - \bar{u}(t_j)}{h_k}\right\|^2 dt \\
=& \int_0^T \left\|\frac{u^k(t_{j+1}) - \bar{u}(t_{j+1})}{h_k} - \frac{u^k(t_j) - \bar{u}(t_j)}{h_k}\right\|^2 dt \\
=& \int_0^T \left\|\frac{x^k(t_{j+1}) - \bar{x}(t_{j+1})}{h_k} - \frac{x^k(t_j) - \bar{x}(t_j)}{h_k}\right\|^2 dt \\
=& \int_0^T \left\|\frac{x^k(t_{j+1}) - x^k(t_j)}{h_k} - \frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{h_k}\right\|^2 dt \\
=& \int_0^T \left\|\dot{x}^k(t) - w_1^k(t)\right\|^2 dt \to 0 \ \text{ as } \ k \to \infty
\end{aligned}$$

due to the above convergence of $\{\dot{x}^k(\cdot)\}$. This verifies (2.16) and completes the proof of the theorem. $\square$

2.3 Existence of Optimal Sweeping Solutions and Relaxation

In this section we start studying optimal solutions to the original sweeping control problem (P). By taking into account the discussion above and further applications, our main attention is paid to the parametric family of *problems* (P^τ) as $\tau \geq 0$ with $(P^0) = (P)$, which are different from (P) only in that the control constraint (1.9) is replaced by those in (2.9). First we establish the following existence theorem of optimal solutions for (P^τ) in the the class of $W^{1,2}([0,T])$ functions.

Theorem 2.3 (existence of sweeping optimal solutions). *Given $r > 0$ and $T > 0$, consider the optimal control problem (P^τ) for any fixed $\tau \in [0, \overline{\tau}]$ as $\overline{\tau} := \min\{r, T\}$ in the equivalent form of the differential inclusion* (2.8) *over all the $W^{1,2}[0,T]$ triples $z(\cdot) = (x(\cdot), u(\cdot), a(\cdot))$. In addition to the assumptions in* (H1) *and* (H2)*, suppose that along some minimizing sequence of $z^k(\cdot) = (x^k(\cdot), u^k(\cdot), a^k(\cdot))$, $k \in I\!N$, we have that $\{\dot{u}^k(\cdot)\}$ is bounded in $L^2([0,T]; \mathbb{R}^n)$ while $\{a^k(\cdot)\}$ is bounded in $W^{1,2}([0,T]; \mathbb{R}^d)$ and that the running cost ℓ in* (1.5) *is convex with respect to the velocity variables $(\dot{x}, \dot{u}, \dot{a})$. Then each sweeping control problem (P^τ) admits an optimal solution.*

Proof. Fix any $\tau \in [0, \overline{\tau}]$ and deduce from Proposition (2.1) that the set of feasible solutions to (P^τ) is nonempty. It follows from the assumption imposed on $\{(u^k(\cdot), a^k(\cdot))\}$ by basic functional analysis that the sequence $\{(\dot{u}^k(\cdot), \dot{a}^k(\cdot))\}$ is weakly compact in $L^2([0,T]; \mathbb{R}^{n+d})$. Thus there are functions $\vartheta^u(\cdot) \in L^2([0,T]; \mathbb{R}^n)$ and $\vartheta^a(\cdot) \in L^2([0,T]; \mathbb{R}^d)$ such that $\dot{u}^k(\cdot) \to \vartheta^u(\cdot)$ and $\dot{a}^k(\cdot) \to \vartheta^a(\cdot)$ along some subsequence $k \to \infty$ weakly in $L^2([0,T]; \mathbb{R}^n)$ and $L^2([0,T]; \mathbb{R}^d)$, respectively. By taking into account that $\|u^k(0)\| = r$ by (2.9) and that the sequence $\{a^k(0)\}$ is bounded, we can assume without loss of generality that $u^k(0) \to u_0$ and $a^k(0) \to a_0$ as $k \to \infty$ for some $u_0 \in \mathbb{R}^n$ and $a_0 \in \mathbb{R}^d$. Defining now the absolutely continuous functions $\bar{u} : [0,T] \to \mathbb{R}^n$ and $\bar{a} : [0,T] \to \mathbb{R}^d$ by

$$\bar{u}(t) := u_0 + \int_0^t \vartheta^u(s)ds \ \text{ and } \ \bar{a}(t) := a_0 + \int_0^t \vartheta^a(s)ds, \tag{2.28}$$

we see that $(u^k(\cdot), a^k(\cdot)) \to (\bar{u}(\cdot), \bar{a}(\cdot))$ in the norm of $W^{1,2}([0,T];\mathbb{R}^{n\times d})$. This implies that $(\bar{u}(\cdot), \bar{a}(\cdot)) \in W^{1,2}([0,T];\mathbb{R}^{n\times d})$ and that $\bar{u}(\cdot)$ satisfies the constraints in (2.9). Furthermore, it follows from Proposition (2.1) that the trajectories $x^k(\cdot)$ of (2.8) uniquely generated by $(u^k(\cdot), a^k(\cdot))$ are uniformly bounded in $W^{1,2}([0,T];\mathbb{R}^n)$, and hence a subsequence of them converges to some $\bar{x}(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$.

Let us show that the limiting triple $\bar{z}(\cdot)$ satisfies (2.8) with $F(z)$ defined in (2.5) and that

$$J[\bar{x}, \bar{u}, \bar{a}] \le \liminf_{k\to\infty} J[x^k, u^k, a^k] \tag{2.29}$$

for the cost functional (1.5). To proceed, we apply the Mazur weak closure theorem to the sequence $\{\dot{z}^k(\cdot)\}$, which tells us that the sequence of convex combination of $\dot{z}^k(\cdot)$ converges to $\dot{\bar{z}}(\cdot)$ weakly in $L^2[0,T]$, and so its subsequence converges to $\dot{\bar{z}}(t)$ for a.e. $t \in [0,T]$. It follows from the above that $\bar{z}(\cdot)$ satisfies the differential inclusion (2.8) due to the convexity of the sets $F(z)$. Using finally the imposed convexity of the running cost ℓ in $\dot{z}$ and the assumptions in (H2) together with the Lebesgue dominated convergence theorem yields (2.29) and thus completes the proof of the theorem. □

We can see that the underlying assumption of Theorem (2.3) is the *convexity* of the integrand ℓ with respect to velocities. This assumption, which is not needed for deriving necessary optimality conditions, can be generally relaxed (and even fully dismissed in rather broad nonconvex settings from the viewpoint of actual solving optimization problems for differential inclusions) due to the so-called *Bogoluybov-Young relaxation procedure.* To describe it in the setting of (P^τ), denote by $\ell_F(t,x,u,a,\dot{x},\dot{u},\dot{a})$ the convexification of the integrand in (1.5) on the set $F(x,u,a)$ from (2.5) with respect to the velocity variables $(\dot{x},\dot{u},\dot{a})$ for all t,x,u,a, i.e., the largest convex and l.s.c. function majorized by $\ell(t,x,u,a,\cdot,\cdot,\cdot)$ on this set; we put $\widehat{\ell}:=\infty$ at points out of $F(x,u,a)$. Define now the *relaxed sweeping problem* (R^τ) by

$$\text{minimize}\ \ \widehat{J}[z]:=\varphi\big(x(T)\big)+\int_0^T\widehat{\ell}_F\big(t,x(t),u(t),a(t),\dot{x}(t),\dot{u}(t),\dot{a}(t)\big)dt \tag{2.30}$$

over all the triples $z(\cdot)=(x(\cdot),u(\cdot),a(\cdot))\in W^{1,2}[0,T]$ satisfying the constraints in (2.9). Of course, there is no difference between problems (P^τ) and (R^τ) if the integrand ℓ is convex with respect to $(\dot{x},\dot{u},\dot{a})$. Furthermore, Theorem (2.3) ensures the existence of optimal solutions to (R^τ). The strong relationship between the original and relax/convexified problems, known as *relaxation stability*, is that in many situations the optimal values of the cost functionals therein agree. This phenomenon has been well recognized for differential inclusions with Lipschitzian right-hand sides in state variables (see [49]), which is never the case for the sweeping process. A more subtle result of this type is obtained in [19, Theorem 4.2] for differential inclusions satisfying the modified one-sided Lipschitz property, which however is also restrictive in applications to sweeping control. The relaxation stability result that directly concerns sweeping control problems is given in [20, Theorem 2] while it deals only with the case of controlled perturbations. In general, relaxation stability in sweeping optimal control is an open question.

Our current study here and its continuation in [10] concern local optimal solutions to (P^τ) involving a local version of relaxation stability. Following [35], we say that $\bar{z}(\cdot)$ is a *relaxed intermediate local minimizer* (r.i.l.m) for (P^τ) if it is feasible to this problem with $J[\bar{z}]=\widehat{J}[\bar{z}]$ and if there are numbers $\alpha\geq 0$ and $\epsilon>0$ such that $J[\bar{z}]\leq J[z]$ for any feasible solution $z(\cdot)$ to (P^τ) satisfying

$$\|z(t)-\bar{z}(t)\|<\epsilon \;\text{ for all }\; t\in[0,T] \;\text{ and }\; \alpha\int_0^T\|\dot{z}(t)-\dot{\bar{z}}(t)\|^2\,dt<\epsilon. \tag{2.31}$$

This notion distinguishes local minimizers that lie between classical weak and strong minima in continuous-time variational problems and can be strictly different from both of them even in fully convex settings; see [37] for discussions, examples, and references. It is clear that from the viewpoint of deriving necessary optimality conditions we can confine ourselves to the case of $\alpha=1$.

2.4 Discrete Approximations of Local Optimal Solutions

In this section we construct a sequence of well-posed discrete approximations of each problem (P^τ) as $0\le\tau\le\overline{\tau}$ with $\overline{\tau}=\min\{r,T\}$ and then employ this method to the study of relaxed intermediate local minimizers for this problem. Given any r.i.l.m. $\bar{z}=(\bar{x}(\cdot),\bar{u}(\cdot),\bar{a}(\cdot))$ for (P^τ) and the discrete mesh Δ_k from (2.10), for every $k\in I\!N$ define the *discrete sweeping control problem* (P_k^τ) as follows: minimize

$$\begin{aligned}
J_k[z^k]:=\varphi(x_k^k)+h_k\sum_{j=0}^{k-1}\ell\Big(t_j^k,x_j^k,u_j^k,a_j^k,\frac{x_{j+1}^k-x_j^k}{h_k},\frac{u_{j+1}^k-u_j^k}{h_k},\frac{a_{j+1}^k-a_j^k}{h_k}\Big)\\
+\sum_{j=0}^{k-1}\int_{t_j^k}^{t_{j+1}^k}\bigg(\Big\|\frac{x_{j+1}^k-x_j^k}{h_k}-\dot{\bar{x}}(t)\Big\|^2+\Big\|\frac{u_{j+1}^k-u_j^k}{h_k}-\dot{\bar{u}}(t)\Big\|^2+\Big\|\frac{a_{j+1}^k-a_j^k}{h_k}-\dot{\bar{a}}(t)\Big\|^2\bigg)dt\\
+\text{dist}^2\bigg(\Big\|\frac{u_1^k-u_0^k}{h_k}\Big\|;(-\infty,\widetilde{\mu}]\bigg)+\text{dist}^2\bigg(\sum_{j=0}^{k-2}\Big\|\frac{u_{j+2}^k-2u_{j+1}^k+u_j^k}{h_k}\Big\|;(-\infty,\widetilde{\mu}]\bigg)\\
+\Big\|\frac{x_1^k-x_0^k}{h_k}-\dot{\bar{x}}(0)\Big\|^2
\end{aligned} \tag{2.32}$$

over elements $z^k := (x_0^k, x_1^k, \ldots, x_k^k, u_0^k, u_1^k, \ldots, u_{k-1}^k, a_0^k, a_1^k, \ldots, a_{k-1}^k)$ satisfying the constraints

$$x_{j+1}^k \in x_j^k - h_k F(x_j^k, u_j^k, a_j^k) \text{ for } j = 0, \ldots, k-1 \text{ with } (x_0^k, u_0^k, a_0^k) = \big(x_0, \bar{u}(0), \bar{a}(0)\big), \tag{2.33}$$

$$\langle x_i^*, x_k^k - u_k^k \rangle \leq 0 \text{ for } i = 1, \ldots, m, \tag{2.34}$$

$$\begin{cases} \|u_j^k\| = r \text{ for } j = j_\tau(k), \ldots, j^\tau(k); \\ r - \tau - \varepsilon_k \leq \|u_j^k\| \leq r + \tau + \varepsilon_k \text{ for } j \leq j_\tau(k) - 1 \text{ and } j \geq j^\tau(k) + 1, \end{cases} \tag{2.35}$$

$$\big\|(x_j^k, u_j^k, a_j^k) - \big(\bar{x}(t_j^k), \bar{u}(t_j^k), \bar{a}(t_j^k)\big)\big\| \leq \epsilon/2 \text{ for } j = 0, \ldots, k-1, \tag{2.36}$$

$$\begin{aligned} \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Bigg(\left\| \frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t) \right\|^2 + \left\| \frac{u_{j+1}^k - u_j^k}{h_k} - \dot{\bar{u}}(t) \right\|^2 \\ + \left\| \frac{a_{j+1}^k - a_j^k}{h_k} - \dot{\bar{a}}(t) \right\|^2 \Bigg) dt \leq \frac{\epsilon}{2}, \end{aligned} \tag{2.37}$$

$$\left\| \frac{u_1^k - u_0^k}{t_1^k - t_0^k} \right\| \leq \widetilde{\mu} + 1, \text{ and } \sum_{j=0}^{k-2} \left\| \frac{u_{j+2}^k - 2u_{j+1}^k + u_j^k}{h_k} \right\| \leq \widetilde{\mu} + 1, \tag{2.38}$$

where $\epsilon > 0$ is taken from definition (2.31) with $\alpha = 1$ while ε_k and $\widetilde{\mu}$ are taken from Theorem (2.2).

Let us first show that each problem (P_k^τ) admits an optimal solution for all large $k \in I\!\!N$; this issue is unavoidable in employing the method of discrete approximations to study local minimizers for (P^τ).

Proposition 2.4 (existence of optimal solutions to discrete approximations). *Suppose that* (H1) *holds and that* (H2) *is also satisfied around the given local minimizer* $\bar{z}(\cdot)$ *for* (P^τ). *Then each problem* (P_k^τ) *admits an optimal solution provided that* $k \in I\!\!N$ *is sufficiently large.*

Proof. Theorem (2.2) tells us that the set of feasible solutions to (P_k^τ) is nonempty for all large $k \in I\!N$. Moreover, the constraints in (2.35)–(2.37) ensure that this set is bounded. To justify the claimed existence of optimal solutions to (P_k^τ) by the Weierstrass existence theorem, it remains to verify that this set is closed. To proceed, take a sequence $z^\nu(\cdot) = z^\nu := (x_0^\nu, \ldots, x_k^\nu, u_0^\nu, \ldots, u_{k-1}^\nu, a_0^\nu, \ldots, a_{k-1}^\nu)$ of feasible solutions for (P_k^τ) converging to some $z(\cdot) = z := (x_0, \ldots, x_k, u_0, \ldots, u_{k-1}, a_0, \ldots, a_{k-1})$ as $\nu \to \infty$ and show that z is feasible to (P_k^τ) as well. Observe that $\langle x_i^*, x_j - u_j \rangle = \lim_{\nu\to\infty} \langle x_i^*, x_j^\nu - u_j^\nu \rangle \le 0$ for all $i = 1, \ldots, m$, and $j = 0, \ldots, k-1$, and so $x_j - u_j \in C$ for all $j = 0, \ldots, k-1$. Picking now $i \in \{1, \ldots, m\} \setminus I(x_j - u_j)$, we have $\langle x_i^*, x_j - u_j \rangle < 0$, which yields $\langle x_i^*, x_j^\nu - u_j^\nu \rangle < 0$ for ν sufficiently large. Then it follows that $i \in \{1, \ldots, m\} \setminus I(x_j^\nu - u_j^\nu)$ and hence $I(x_j^\nu - u_j^\nu) \subset I(x_j - u_j)$ for $\nu \in I\!N$ sufficiently large. By taking (2.5) and (2.7) into account, we get the equalities

$$x_{j+1}^\nu - x_j^\nu = -h_k \Big(\sum_{i \in I(x_j^\nu - u_j^\nu)} \lambda_{ji}^\nu x_i^* + f(x_j^\nu, a_j^\nu) \Big) = -h_k \Big(\sum_{i \in I(x_j - u_j)} \lambda_{ji}^\nu x_i^* + f(x_j^\nu, a_j^\nu) \Big),$$

where $\lambda_{ji}^\nu := 0$ if $i \in I(x_j - u_j) \setminus I(x_j^\nu - u_j^\nu)$. This shows therefore that

$$\frac{x_{j+1}^\nu - x_j^\nu}{-h_k} - f(x_j^\nu, a_j^\nu) = \sum_{i \in I(x_j - u_j)} \lambda_j^\nu x_i^* \in N(x_j - u_j; C).$$

Passing there to the limit as $\nu \to \infty$ and using the closedness of $N(x_j - u_j; C)$ give us

$$\frac{x_{j+1} - x_j}{-h_k} - f(x_j, a_j) \in N(x_j - u_j; C),$$

which ensures that $x_{j+1} \in x_j - h_k F(x_j, u_j, a_j)$ and thus completes the proof of the proposition.

□

The next theorem is a key result of the method of discrete approximations in sweeping optimal control. It shows that optimal solutions to (P^τ) and (P_k^τ) are so closely related that solving the continuous-time control problem (P^τ) for small $\tau \ge 0$ can be practically replaced by solving its finite-dimensional discrete counterparts (P_k^τ) when k is sufficiently large. Moreover, it justifies the possibility to derive necessary optimality conditions for local minimizers of (P^τ) by passing to the limit from those in (P_k^τ) as $k \to \infty$.

Theorem 2.5 (strong discrete approximation of intermediate local minimizers). *Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot))$ be a r.i.l.m. for problem (P^τ), where $\tau \in [0, \overline{\tau}]$ with $\overline{\tau} = \min\{r, T\}$. In addition to the assumptions in Theorem (2.2) and Proposition (2.4) imposed on $\bar{z}(\cdot)$, suppose that both terminal and running costs in (1.5) are continuous at $\bar{x}(T)$ and at $(t, \bar{z}(t), \dot{\bar{z}}(t))$ for a.e. $t \in [0, T]$, respectively, and that $\ell(\cdot, z, \dot{z})$ is uniformly majorized by a summable function near the given local minimizer. Then any sequence of piecewise linearly extended to $[0, T]$ optimal solutions $\bar{z}^k(\cdot) = (\bar{x}^k(\cdot), \bar{u}^k(\cdot), \bar{a}^k(\cdot))$ of (P_k^τ) converges to $\bar{z}(\cdot)$ in the norm topology of $W^{1,2}([0, T]; \mathbb{R}^{2n+d})$ with*

$$\frac{\bar{x}_1^k - \bar{x}_0^k}{h_k} \to \dot{\bar{x}}(0) \quad as \quad k \to \infty \tag{2.39}$$

and the validity of the estimates

$$\left\| \frac{\bar{u}_1^k - \bar{u}_0^k}{h_k} \right\| \le \widetilde{\mu}, \quad \limsup_{k\to\infty} \sum_{j=0}^{k-2} \left\| \frac{\bar{u}_{j+2}^k - 2\bar{u}_{j+1}^k + \bar{u}_j^k}{h_k} \right\| \le \widetilde{\mu}, \tag{2.40}$$

where the number $\widetilde{\mu}$ is calculated in (2.17).

Proof. Fix a sequence of optimal solutions $\bar{z}^k(\cdot)$ to (P_k^τ), which exists by Proposition (2.4). It is easy to see that all the statements of the theorem are implied by the equality

$$\begin{aligned} \lim_{k\to\infty} \int_0^T & \Big(\big\|\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t)\big\|^2 + \big\|\dot{\bar{u}}(t) - \dot{\bar{u}}^k(t)\big\|^2 + \big\|\dot{\bar{a}}(t) - \dot{\bar{a}}^k(t)\big\|^2 \Big) dt \\ & + \left\| \frac{\bar{x}_1^k - \bar{x}_0^k}{h_k} - \dot{\bar{x}}(0) \right\|^2 + \mathrm{dist}^2\left(\left\| \frac{\bar{u}_1^k - \bar{u}_0^k}{h_k} \right\|; (-\infty, \widetilde{\mu}] \right) \\ & + \mathrm{dist}^2\left(\sum_{j=0}^{k-2} \left\| \frac{\bar{u}_{j+2}^k - 2\bar{u}_{j+1}^k + \bar{u}_j^k}{h_k} \right\|; (-\infty, \widetilde{\mu}] \right) = 0. \end{aligned} \tag{2.41}$$

To justify (2.41), suppose that it does not hold, i.e., there is a subsequence of natural numbers (without relabeling) along which the limit in (2.41) equals to some $c > 0$. By the weak compactness of the unit ball in $L^2([0,T];\mathbb{R}^{2n+d})$ we can find a triple $(v(\cdot),w(\cdot),q(\cdot)) \in L^2([0,T];\mathbb{R}^{2n+d})$ and yet another subsequence of $\{z^k(\cdot)\}$–again without relabeling–such that

$$\big(\dot{\bar{x}}^k(\cdot),\dot{\bar{u}}^k(\cdot),\dot{\bar{a}}^k(\cdot)\big) \to \big(v(\cdot),w(\cdot),q(\cdot)\big) \text{ weakly in } L^2([0,T];\mathbb{R}^{2n+d}).$$

Define now the absolutely continuous function $\widetilde{z}(\cdot) := (\widetilde{x}(\cdot),\widetilde{u}(\cdot),\widetilde{a}(\cdot))\colon [0,T] \to \mathbb{R}^{2n+d}$ by

$$\widetilde{z}(t) := \big(x_0,\bar{u}(0),\bar{a}(0)\big) + \int_0^t \big(v(s),w(s),q(s)\big)ds, \quad t \in [0,T],$$

which gives us $\dot{\widetilde{z}}(t) = (v(t),w(t),q(t))$ a.e. on $[0,T]$ and implies that $\dot{\bar{z}}^k(\cdot) \to \dot{\widetilde{z}}(\cdot) = (\dot{\widetilde{x}}(\cdot),\dot{\widetilde{u}}(\cdot),\dot{\widetilde{a}}(\cdot))$ weakly in $L^2([0,T];\mathbb{R}^{2n+d})$ and therefore $\widetilde{z}(\cdot) \in W^{1,2}([0,T];\mathbb{R}^{2n+d})$. As follows from the Mazur weak closure theorem, there exists a sequence of convex combinations of $\dot{\bar{z}}^k(\cdot)$ that converges to $\dot{\widetilde{z}}(\cdot)$ strongly in $L^2([0,T];\mathbb{R}^{2n+d})$ and thus almost everywhere on $[0,T]$ along a subsequence. It is clear that the limiting u-component $\widetilde{u}(\cdot)$ obeys the constraints in (2.9). Let us verify that $\widetilde{z}(\cdot)$ satisfies the differential inclusion (1.6), where the moving set $C(t)$ is generated by $\widetilde{u}(\cdot)$ in (1.7).

To proceed, observe first that $\widetilde{x}(t) - \widetilde{u}(t) = \lim_{k\to\infty}(\bar{x}^k(t) - \bar{u}^k(t)) \in C$ by the closedness of the polyhedron C. It follows from the above that there are a function $\nu : I\!N \to I\!N$ and a sequence of real numbers $\{\alpha(k)_j |\ j = k,\ldots,\nu(k)\}$ such that

$$\alpha(k)_j \geq 0,\ \sum_{j=k}^{\nu(k)} \alpha(k)_j = 1, \ \text{ and } \ \sum_{j=k}^{\nu(k)} \alpha(k)_j \dot{\bar{z}}^j(t) \to \dot{\widetilde{z}}(t) \ \text{ a.e. } \ t \in [0,T]$$

as $k \to \infty$. Then by the closedness and convexity of the normal cone we have the relationships

$$-\dot{\widetilde{x}}(t) - f\big(\widetilde{x}(t), \widetilde{a}(t)\big) = \lim_{k\to\infty}\left(-\sum_{j=k}^{\nu(k)} \alpha(k)_j \dot{\bar{x}}^j(t) - \sum_{j=k}^{\nu(k)} \alpha(k)_j f\big(\bar{x}^j(t), \bar{a}^j(t)\big)\right)$$

$$= \lim_{k\to\infty} \sum_{i\in I(\widetilde{x}(t)-\widetilde{u}(t))} \left(\sum_{j=k}^{\nu(k)} \alpha(k)_j \lambda_i^j\right) x_i^* \in N\big(\widetilde{x}(t) - \widetilde{u}(t); C\big) \ \text{ a.e. } \ t \in [0,T],$$

where $I(\cdot)$ is taken from (2.6), and where $\lambda_i^j = 0$ if $i \in I(\widetilde{x}(t) - \widetilde{u}(t)) \backslash I(x^j(t) - u^j(t))$ for $j = k, \ldots, \nu(k)$ and all large $k \in I\!N$. It shows by (2.7) that $\widetilde{z}(\cdot)$ satisfies (2.8) and hence the constraints in (1.8).

Consider further the integral functional

$$\mathcal{I}[y] := \int_0^T \|y(t) - \dot{\bar{z}}(t)\|^2 \, dt$$

is l.s.c. in the weak topology of $L^2([0,T]; \mathbb{R}^{2n+d})$ due to the convexity of the integrand in y. Hence

$$\mathcal{I}(\dot{\widetilde{z}}) = \int_0^T \left\|\dot{\widetilde{z}}(t) - \dot{\bar{z}}(t)\right\|^2 dt \le \liminf_{k\to\infty} \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \left\| \frac{\bar{z}_{j+1}^k - \bar{z}_j^k}{h_k} - \dot{\bar{z}}(t) \right\|^2 dt \tag{2.42}$$

by the construction of $\widetilde{z}(\cdot)$. Passing to the limit in (2.36) and (2.37) as $k \to \infty$ and using (2.42), we get

$$\|\widetilde{z}(t) - \bar{z}(t)\| \le \epsilon/2 \ \text{ on } \ [0,T] \ \text{ and } \ \int_0^T \left\|\dot{\widetilde{z}}(t) - \dot{\bar{z}}(t)\right\|^2 dt \le \epsilon/2.$$

This means that $\widetilde{z}(\cdot)$ belongs to the given neighborhood of $\bar{z}(\cdot)$ in $W^{1,2}([0,T]; \mathbb{R}^{2n+d})$. Furthermore, the definition of ℓ_F in (2.30) and its convexity in the velocity variables yield

$$\int_0^T \widehat{\ell}_F\big(t, \widetilde{x}(t), \widetilde{u}(t), \widetilde{a}(t), \dot{\widetilde{x}}(t), \dot{\widetilde{u}}(t), \dot{\widetilde{a}}(t)\big) dt$$

$$\le \liminf_{k\to\infty} h_k \sum_{j=0}^{k-1} \ell\left(t_j^k, \bar{x}_j^k, \bar{u}_j^k, \bar{a}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}, \frac{\bar{u}_{j+1}^k - \bar{u}_j^k}{h_k}. \frac{\bar{a}_{j+1}^k - \bar{a}_j^k}{h_k}\right).$$

Thus the passage to the limit in the cost functional of (P_k^τ) and the assumption on $c > 0$ in the negation of (2.41) together with (H2) bring us to the relationships

$$\widehat{J}[\widetilde{z}]+c=\varphi\big(\widetilde{x}(T)\big)+\int_0^T \widehat{\ell}_F\big(t,\widetilde{x}(t),\widetilde{u}(t),\widetilde{a}(t),\dot{\widetilde{x}}(t),\dot{\widetilde{u}}(t),\dot{\widetilde{a}}(t)\big)dt+c\le \liminf_{k\to\infty} J_k[\bar{z}^k]. \quad (2.43)$$

Applying now Theorem (2.2) to the r.i.l.m. $\bar{z}(\cdot)$ gives us a sequence $\{z^k(\cdot)\}$ of feasible solutions to (P_k^τ) whose extensions to the whole interval $[0,T]$ strongly approximate $\bar{z}(\cdot)$ in the $W^{1,2}$ topology with the additional convergence in (2.13). Since $\bar{z}^k(\cdot)$ is an optimal solution to (P_k^τ), we have

$$J_k[\bar{z}^k]\le J_k[z^k]\ \text{ for each }\ k\in I\!N. \quad (2.44)$$

It follows from the structure of the cost functionals in (P_k^τ), the strong $W^{1,2}$-convergence of $z^k(\cdot)\to\bar{z}(\cdot)$ together with (2.13) in Theorem (2.2), and the continuity assumptions on φ and ℓ imposed in this theorem that $J_k[z^k]\to J[\bar{z}]$ as $k\to\infty$. Then passing to the limit in (2.44) gives us

$$\limsup_{k\to\infty} J_k[\bar{z}^k]\le J[\bar{z}]. \quad (2.45)$$

Combining finally (2.43) and (2.45) with $c>0$ and the definition of r.i.l.m., we get

$$\widehat{J}[\widetilde{z}]+c\le J[\bar{z}]=\widehat{J}[\bar{z}],\ \text{ and so }\ \widehat{J}[\widetilde{z}]<\widehat{J}[\bar{z}],$$

which clearly contradicts the fact that $\bar{z}(\cdot)$ is a r.i.l.m. for problem (P^τ). This justifies the validity of (2.41) and thus completes the proof of the theorem. □

2.5 Generalized Differentiation and Calculations

After establishing close connections between optimal solutions to the original and discretized sweeping control problems, our further goal is to derive effective necessary optimality conditions to each problem (P_k^τ) defined in (2.32)–(2.38). Looking at this problem, we can see that it is *intrinsically nonsmooth*, even if both terminal and running costs are assumed to be differentiable, which is not the case here. The main reason for this is the unavoidable presence of the geometric constraints (2.33) with F given by (2.5) whose increasing number comes from the discretization of the sweeping differential inclusion (1.6). We can deal with such problems by using the robust generalized differential constructions, which are basic in variational analysis and its applications; see, e.g., the books [6, 36, 45] and the references therein. Here we first recall their definitions with a brief overview of the needed properties and then deduce from [22] major coderivative calculations for the mapping F in (2.5) via the initial data of the sweeping process. This together with available calculus rules of generalized differentiation plays a crucial role in deriving verifiable necessary optimality conditions for the sweeping control problems under consideration.

Given a set-valued mapping/multifunction $G\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, denote by

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} G(x) := \Big\{ y\in\mathbb{R}^m \Big|\ \exists\ \text{ sequences }\ x_k\to\bar{x},\ y_k\to y\ \text{ such that } \\ y_k\in G(x_k),\ k\in I\!N \Big\} \tag{2.46}$$

the (Kuratowski-Painlevé) *outer limit* of G at $\bar{x}$ with $G(\bar{x})\neq\emptyset$. Considering now a set $\Omega\subset\mathbb{R}^n$ locally closed around $\bar{x}\in\Omega$, the (Mordukhovich basic/limiting) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := N_\Omega(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} \{\mathrm{cone}\,[x-\Pi(x;\Omega)]\} \tag{2.47}$$

via the outer limit (2.46), where $\Pi(x;\Omega)$ stands for the Euclidean projection of x onto Ω. When Ω is convex, (2.47) reduces to the normal cone of convex analysis, but it is often nonconvex in nonconvex settings. The crucial feature of (2.47) and the associated subdifferential and coderivative constructions for functions and multifunctions (see below) is *full calculus* based on variational and extremal principles.

For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with its graph

$$\operatorname{gph} F := \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \big|\ y \in F(x)\big\}$$

locally closed around $(\bar{x}, \bar{y})$, the *coderivative* of F at $(\bar{x}, \bar{y})$ generated by (2.47) is defined by

$$D^*F(\bar{x}, \bar{y})(u) := \big\{v \in \mathbb{R}^n \big|\ (v, -u) \in N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big)\big\},\ u \in \mathbb{R}^m. \tag{2.48}$$

When $F : \mathbb{R}^n \to \mathbb{R}^m$ is single-valued and continuously differentiable ($\mathcal{C}^1$) around $\bar{x}$, we have

$$D^*F(\bar{x})(u) = \big\{\nabla F(\bar{x})^* u\big\} \text{ for all } u \in \mathbb{R}^m$$

via the adjoint/transposed Jacobian matrix $\nabla F(\bar{x})^*$, where $\bar{y} = F(\bar{x})$ is omitted.

Given an extended-real-valued l.s.c. function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ with its domain and epigraph

$$\operatorname{dom} \varphi := \big\{x \in \mathbb{R}^n \big|\ \varphi(x) < \infty\big\} \text{ and } \operatorname{epi} \varphi := \big\{(x, \alpha) \in \mathbb{R}^{n+1} \big|\ \alpha \geq \varphi(x)\big\},$$

the (first-order) *subdifferential* of φ at $\bar{x} \in \operatorname{dom} \varphi$ is generated geometrically by (2.47) as

$$\partial\varphi(\bar{x}) := \big\{v \in \mathbb{R}^m \mid (v, -1) \in N\big((\bar{x}, \varphi(\bar{x})); \operatorname{epi} \varphi\big)\big\}$$

while admitting various equivalent analytic representations that can be found, e.g., in the books [36, 45].

Our main emphases here is on evaluating the coderivative of the set-valued mapping F from (2.5) entirely via the given data of the perturbed sweeping process. Note that the partial normal cone structure of the mapping F reveals the *second-order* subdifferential nature of the aforementioned construction in the sense of [34]. For simplicity in further applications, suppose below that the perturbation function f is smooth while observing that the available calculus rules allow us to consider Lipschitzian perturbations.

Having in mind representation (2.8) of the mapping F in terms of the generating vectors x_i^* of the convex polyhedron (1.7) with the active constraint indices $I(\bar{x})$ in (2.7), consider the following subsets:

$$I_0(y) := \big\{i \in I(\bar{x})\big|\ \langle x_i^*, y\rangle = 0\big\} \ \text{ and } \ I_>(y) := \big\{i \in I(\bar{x})\big|\ \langle x_i^*, y\rangle > 0\big\}, \quad y \in \mathbb{R}^n. \tag{2.49}$$

The next theorem provides an effective upper estimate of the coderivative of F with ensuring the equality therein under an additional assumption on x_i^*.

Theorem 2.6 (calculating the coderivative of the sweeping control mapping). *Given F in (2.5) with C from (1.7), assume that f is smooth and denote $G(x) := N(x; C)$. Then for any $(x, u, a) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ and $w - f(x, a) \in G(x - u)$ we have the coderivative upper estimate*

$$\begin{aligned} D^*F(x,u,a,w)(y) \subset \Bigg\{ z^* \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \Bigg| \ z^* = \Bigg(\nabla_x f(x,a)^* y + \sum_{i \in I_0(y) \cup I_>(y)} \gamma_i x_i^*, \\ - \sum_{i \in I_0(y) \cup I_>(y)} \gamma_i x_i^*, \nabla_a f(x,a)^* y\Bigg)\Bigg\}, \quad y \in \operatorname{dom} D^*G\big(x-u, w-f(x,a)\big), \end{aligned} \tag{2.50}$$

where $I_0(y)$ and $I_>(y)$ are defined in (2.49) with $\bar{x} = x - u$, and where $\gamma_i \in \mathbb{R}$ for $i \in I_0(y)$ while $\gamma_i \geq 0$ for $i \in I_>(y)$. Furthermore, (2.50) holds as an equality and the domain $\operatorname{dom} D^*G\big(x-u, w-f(x,a)\big)$ *can be computed by*

$$\begin{aligned} \operatorname{dom} D^*G\big(x-u, w-f(x,a)\big) = \Bigg\{ y \big| \ \exists\, \lambda_i \geq 0 \text{ such that } w - f(x,a) = \sum_{i \in I(x-u)} \lambda_i x_i^*, \\ \lambda_i > 0 \Longrightarrow \langle x_i^*, y\rangle = 0 \Bigg\} \end{aligned} \tag{2.51}$$

provided that the generating vectors $\{x_i^* |\ i \in I(x-u)\}$ of the polyhedron C are linearly independent.

Proof. Picking any $y \in \operatorname{dom} D^*G(x-u, w-f(x,a))$ and $z^* \in D^*F(x,u,a,X)(y)$ and then denoting $\widetilde{G}(x,u,a) := G(x-u)$ and $\widetilde{f}(x,u,a) := f(x,a)$, we deduce from [37, Theorem 1.62] that

$$z^* \in \nabla \widetilde{f}(x,u,a)^* y + D^*\widetilde{G}\big(x,u,a,w-f(x,a)\big)(y).$$

Observe then the obvious representation

$$\widetilde{G}(x,u,a) = G \circ g(x,u,a) \;\text{ with }\; g(x,u,a) := x-u,$$

where the latter mapping has the surjective derivative. It follows from [37, Theorem 1.66] that

$$z^* \in \nabla \widetilde{f}(x,u,a)^* y + \nabla g(x,u,a)^* D^*G\big(x-u, w-f(x,a)\big)(y). \tag{2.52}$$

Employing now in (2.52) the coderivative estimate for the normal cone mapping G obtained in [22, Theorem 4.5] with the exact coderivative calculation given in [22, Theorem 4.6] under the linear independence of the generating vectors x_i^* and also taking into account the structure of the mapping $\widetilde{f}$ in (2.52), we arrive at (2.50) and the equality therein under the aforementioned assumption. □

2.6 Necessary Optimality Conditions

In this section we derive necessary conditions for optimal solutions to each discrete approximation problems (P_k^τ) with $k \in I\!N$ and $0 \le \tau \le \overline{\tau} = \min\{r, T\}$. As shown in Theorem (2.5), for large $k \in I\!N$ and any fixed $\tau \in [0, \overline{\tau}]$ the constructed optimal solutions $\bar{z}^k(\cdot)$ to (P_k^τ) are practically undistinguished (in the $W^{1,2}$ norm) from the optimal solution $\bar{z}(\cdot)$ to the continuous-time sweeping control problem (P^τ), and so the necessary optimality conditions for $\bar z^k(\cdot)$ obtained below can be well treated as "almost optimality" necessary conditions for the solution $\bar z(\cdot)$ to (P^τ) playing virtually the same role in applications.

To proceed, we first establish necessary optimality conditions for (P^τ_k) in the discrete *Euler-Lagrange form* via the generalized differential constructions of Section 6; cf. [37]. Then employing the complete coderivative calculations of Theorem (2.6) for the underlying mapping F from (2.8) allows us to derive necessary optimality conditions for the sweeping control problem (P^τ_k) entirely in terms of its initial data. Throughout this section we assume that the cost functions φ and ℓ in (2.32) are *locally Lipschitzian* around the points in question and for brevity drop indicating the time-dependence of the running cost ℓ.

Theorem 2.7 (Euler-Lagrange conditions for discrete approximations). *Fixing any $\tau \in [0,\overline{\tau}]$ and $k \in I\!N$, consider an optimal solution $\bar z^k = (x_0, \bar x^k_1, \ldots, \bar x^k_k, \bar u^k_0, \ldots, \bar u^k_k, \bar a^k_0, \ldots, \bar a^k_k)$ to problem (P^τ_k). Then there exist dual elements $\lambda^k \geq 0$, $\alpha^k = (\alpha^k_1, \ldots, \alpha^k_m) \in \mathbb{R}^m_+$, $\xi^k = (\xi^k_0, \ldots, \xi^k_k) \in \mathbb{R}^{k+1}$, and $p^k_j = (p^{xk}_j, p^{uk}_j, p^{ak}_j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ as $j = 0, \ldots, k$ satisfying the conditions*

$$\lambda^k + \|\alpha^k\| + \|\xi^k\| + \sum_{j=0}^{k-1} \|p^{xk}_j\| + \|p^{uk}_0\| + \|p^{ak}_0\| \neq 0, \tag{2.53}$$

$$\alpha^k_i \left\langle x^*_i, \bar x^k_k - \bar u^k_k \right\rangle = 0, \quad i = 1, \ldots, m, \tag{2.54}$$

$$\begin{cases} \xi^k_j(\nu - \|\bar u^k_j\|) \leq 0 \ \ \textit{for all} \ \ \nu \in \big[r - \tau - \varepsilon_k, r + \tau + \varepsilon_k\big] \\ \textit{whenever} \ \ j = 0, \ldots, j_\tau(k) - 1 \ \ \textit{and} \ \ j = j^\tau(k) + 1, \ldots, k. \end{cases} \tag{2.55}$$

$$-p^{xk}_k \in \lambda^k \partial\varphi(\bar x^k_k) + \sum_{i=1}^m \alpha^k_i x^*_i \ p^{uk}_k = \sum_{i=1}^m \alpha^k_i x^*_i - 2\xi^k_k \bar u^k_k, \ p^{ak}_k = 0, \tag{2.56}$$

$$p^{uk}_{j+1} = \lambda^k(v^{uk}_j + h^{-1}_k \theta^{uk}_j), \ p^{ak}_{j+1} = \lambda^k(v^{ak}_j + h^{-1}_k \theta^{ak}_j), \quad j = 0, \ldots, k-1, \tag{2.57}$$

$$\left(\frac{p^{xk}_{j+1}-p^{xk}_j}{h_k}-\lambda^k w^{xk}_j-\frac{\chi^k_j}{h_k},\frac{p^{uk}_{j+1}-p^{uk}_j}{h_k}-\lambda^k w^{uk}_j,\frac{p^{ak}_{j+1}-p^{ak}_j}{h_k}-\lambda^k w^{ak}_j,p^{xk}_{j+1}\right.$$
$$\left.-\lambda^k(v^{xk}_j+h_k^{-1}\theta^{xk}_j)\right)\in\left(0,\frac{2}{h_k}\xi^k_j\bar{u}^k_j,0,0\right)+N\left(\left(\bar{x}^k_j,\bar{u}^k_j,\bar{a}^k_j,\frac{\bar{x}^k_{j+1}-\bar{x}^k_j}{-h_k}\right);\operatorname{gph}F\right) \tag{2.58}$$

for $j=1,\ldots,k-1$ *and with the subgradient vectors*

$$\left(w^{xk}_j,w^{uk}_j,w^{ak}_j,v^{xk}_j,v^{uk}_j,v^{ak}_j\right)\in\partial\ell\left(\bar{z}^k_j,\frac{\bar{z}^k_{j+1}-\bar{z}^k_j}{h_k}\right),\quad j=0,\ldots,k-1, \tag{2.59}$$

where the sequence of $\varepsilon_k\downarrow 0$ *as* $k\to\infty$ *is taken from* (2.23), *where*

$$\chi^k_j:=\begin{cases}(-1)^{j+1}2\left(\dfrac{\bar{x}^k_1-\bar{x}^k_0}{h_k}-\dot{\bar{x}}(0)\right) & \text{if } j=0,1,\\ 0 & \text{otherwise,}\end{cases} \tag{2.60}$$

and where the vector triples $(\theta^{xk}_j,\theta^{uk}_j,\theta^{ak}_j)$ *for each* $j=0,\ldots,k-1$ *are defined by*

$$(\theta^{xk}_j,\theta^{uk}_j,\theta^{ak}_j):=2\int_{t_j}^{t_{j+1}}\left(\frac{\bar{x}^k_{j+1}-\bar{x}^k_j}{h_k}-\dot{\bar{x}}(t),\frac{\bar{u}^k_{j+1}-\bar{u}^k_j}{h_k}-\dot{\bar{u}}(t),\frac{\bar{a}^k_{j+1}-\bar{a}^k_j}{h_k}-\dot{\bar{a}}(t)\right)dt. \tag{2.61}$$

Proof. Let $y:=(x_0,\ldots,x_k,u_0,\ldots,u_k,a_0,\ldots,a_k,X_0,\ldots,X_{k-1},$ $U_0,\ldots,U_{k-1},A_0,\ldots,A_{k-1})$, where the the starting point x_0 is fixed but the other variables depend on k while we omit the upper index "k" for simplicity. Given $\epsilon>0$ in (P^τ_k), define the problem of mathematical programming (MP) by:

$$\text{minimize}\ \ \varphi_0[y]:=\varphi(x_k)+h_k\sum_{j=0}^{k-1}\ell(x_j,u_j,a_j,X_j,U_j,A_j)$$
$$+\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\left\|(X_j,U_j,A_j)-\dot{\bar{z}}(t)\right\|^2dt+\left\|\frac{x^k_1-x^k_0}{h_k}-\dot{\bar{x}}(0)\right\|^2$$
$$+\text{dist}^2\left(\left\|\frac{u^k_1-u^k_0}{h_k}\right\|;(-\infty,\widetilde{\mu}]\right)+\text{dist}^2\left(\sum_{j=0}^{k-2}\|U_{j+1}-U_j\|;\left(-\infty,\widetilde{\mu}\right]\right)$$

subject to finitely many equality, inequality, and geometric constraints

$$
\begin{aligned}
&b_j^x(y) := x_{j+1} - x_j - h_k X_j = 0 \ \text{ for } \ j = 0, \dots, k-1,\\
&b_j^u(y) := u_{j+1} - u_j - h_k U_j = 0 \ \text{ for } \ j = 0, \dots, k-1,\\
&b_j^a(y) := a_{j+1} - a_j - h_k A_j = 0 \ \text{ for } \ j = 0, \dots, k-1,\\
&g_i(y) := \langle x_i^*, x_k - u_k \rangle \le 0 \ \text{ for } \ i = 1, \dots, m\\
&d_j(y) := \|u_j\|^2 - r^2 = 0 \ \text{ for } \ j = j_\tau(k), \dots, j^\tau(k),\\
&y \in \Omega_j := \big\{ y \big| \ r - \tau - \varepsilon_k \le \|u_j\| \le r + \tau + \varepsilon_k \big\} \ \text{ for } \ j = 0, \dots, j_\tau(k) - 1\\
&\qquad \text{and } \ j = j^\tau(k) + 1, \dots, k,\\
&\phi_j(y) := \|(x_j, u_j, a_j) - \bar{z}(t_j)\| - \epsilon/2 \le 0 \ \text{ for } \ j = 0, \dots, k,\\
&\phi_{k+1}(y) := \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big(\|(X_j, U_j, A_j) - \dot{\bar{z}}(t)\|^2 \Big)\, dt - \frac{\epsilon}{2} \le 0,\\
&\phi_{k+2}(y) := \sum_{j=0}^{k-2} \|U_{j+1} - U_j\| \le \widetilde{\mu} + 1, \quad \phi_{k+3}(y) := \|u_1 - u_0\| \le (\widetilde{\mu} + 1)(t_1^k - t_0^k),\\
&y \in \Xi_j := \big\{ y \big| - X_j \in F(x_j, u_j, a_j) \big\} \ \text{ for } \ j = 0, \dots, k-1,\\
&y \in \Xi_k := \big\{ y \big| x_0 \ \text{ is fixed}, \ (u_0, a_0) = \big(\bar{u}(0), \bar{a}(0)\big) \big\},
\end{aligned}
$$

where the number $\widetilde{\mu}$ is is calculated in (2.17). It follows directly from the construction above that problem (MP) is equivalent to (P_k^τ) for any fixed $k \in I\!N$ and $\tau \in [0, \overline{\tau}]$.

Necessary optimality conditions for problem (MP) in terms of the generalized differential tools of Section 6 can be deduced from [37, Theorem 5.24]. We specify them for the optimal solution $\bar{y} = (\bar{z}, \bar{Z})$ to (MP), where $\bar{z} := (\bar{x}_0, \dots, \bar{x}_k, \bar{u}_0, \dots, \bar{u}_k, \bar{a}_0, \dots, \bar{a}_k)$ is generated by the optimal solution $\bar{z}^k$ to (P_k^τ) while $\bar{Z} := (\bar{X}_0, \dots, \bar{X}_{k-1}, \bar{U}_0, \dots, \bar{U}_{k-1}, \bar{A}_0, \dots, \bar{A}_{k-1})$ signifies the discrete "velocity" determined by the constraints $b_j(\bar{y}) = 0$. It follows from Theorem (2.5) that all the inequality constraints in (MP) relating to functions ϕ_j as $j = 0, \ldots, k+2$ are inactive for large k, and so the corresponding multipliers do not appear in the optimality conditions. Thus we find $\lambda \geq 0$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$, $\xi = (\xi_0, \ldots, \xi_k) \in \mathbb{R}^{k+1}$, $p_j = (p^x_j, p^u_j, p^a_j) \in \mathbb{R}^{2n+d}$ as $j = 0, \ldots, k$, and

$$y^*_j = \big(x^*_{0j}, \ldots, x^*_{kj}, u^*_{0j}, \ldots, u^*_{kj}, a^*_{0j}, \ldots, a^*_{kj}, X^*_{0j}, \ldots, X^*_{(k-1)j}, U^*_{0j}, \ldots, U^*_{(k-1)j}, A^*_{0j}, \ldots, A^*_{(k-1)j}\big)$$

for $j = 0, \ldots, k$, which are not zero simultaneously while satisfying (2.55) and the conditions

$$y^*_j \in \begin{cases} N(\bar{y}; \Xi_j) + N(\bar{y}; \Omega_j) & \text{if } \ j \in \big\{0, \ldots, j_\tau(k) - 1\big\} \cup \big\{j^\tau(k) + 1, \ldots, k\big\}, \\ N(\bar{y}; \Xi_j) & \text{if } \ j \in \big\{j_\tau(k), \ldots, j^\tau(k)\big\}, \end{cases} \tag{2.62}$$

$$-y^*_0 - \ldots - y^*_k \in \lambda \partial \varphi_0(\bar{y}) + \sum_{i=1}^m \alpha_i \nabla g_i(\bar{y}) + \sum_{j=j_\tau(k)}^{j^\tau(k)} \xi_j \nabla d_j(\bar{y}) + \sum_{j=0}^{k-1} \nabla b_j(\bar{y})^* p_{j+1}, \tag{2.63}$$

$$\alpha_i g_i(\bar{y}) = 0 \ \text{ for } \ i = 1, \ldots, m. \tag{2.64}$$

Note that the first line in (2.62) comes from applying the normal cone intersection formula from [36, Corollary 3.5] to $\bar{y} \in \Omega_j \cap \Xi_j$ for $j \in \{0, \ldots, j_\tau(k) - 1\} \cup \{j^\tau(k) + 1, \ldots, k\}$, where the qualification condition imposed therein can be easily verified. It follows from the structure of the sets Ω_j and Ξ_j that (2.55) holds while the inclusions in (2.62) are equivalent to

$$\begin{cases} \left(x^*_{jj}, u^*_{jj}, a^*_{jj}, -X^*_{jj}\right) \in N\left(\left(\bar{x}^k_j, \bar{u}^k_j, \bar{a}^k_j, \dfrac{\bar{x}^k_{j+1} - \bar{x}^k_j}{-h_k}\right); \text{gph}\, F\right), \\ \text{for } j = j_\tau(k), \ldots, j^\tau(k), \\ \left(x^*_{jj}, u^*_{jj} - \psi^u_j, a^*_{jj}, -X^*_{jj}\right) \in N\left(\left(\bar{x}^k_j, \bar{u}^k_j, \bar{a}^k_j, \dfrac{\bar{x}^k_{j+1} - \bar{x}^k_j}{-h_k}\right); \text{gph}\, F\right), \\ \text{for } j \notin \{j_\tau(k), \ldots, j^\tau(k)\} \end{cases} \tag{2.65}$$

with every other components of y^*_j equal to zero, where $\psi^u_j \in N(\bar{u}_j, h^{-1}([r - \tau - \varepsilon_k, r + \tau + \varepsilon_k]))$

and $h(u) := \|u\|^2$. It then follows from [36, Theorem 1.17] that

$$N(\bar u_j, h^{-1}([r-\tau-\varepsilon_k, r+\tau+\varepsilon_k])) = 2\bar u_j N(\|\bar u_j\|; [r-\tau-\varepsilon_k, r+\tau+\varepsilon_k]).$$

Hence $\psi_j^u = 2\xi_j \bar u_j$, where ξ_j satisfies (2.55) and $j \notin \{j_\tau(k), \ldots, j^\tau(k)\}$. Similarly we conclude that the triple $(x_k^*(0), u_k^*(0), a_k^*(0))$ determined by the normal cone to Ξ_k is the only potential nonzero component of y_k^*. This shows that

$$\begin{aligned} -y_0^* - y_1^* - \ldots - y_k^* = \big(&-x_{0k}^* - x_{00}^*, -x_{11}^*, \ldots, -x_{k-1,k-1}^*, 0, -u_{0k}^* - u_{00}^*, \ldots, \\ &-u_{k-1,k-1}^*, 0, -a_{0k}^* - a_{00}^*, -a_{11}^*, \ldots, -a_{k-1,k-1}^*, 0, -X_{00}^*, \ldots, -X_{k-1,k-1}^*, 0, \ldots, 0\big). \end{aligned} \tag{2.66}$$

Next we calculate the three sums on the right-hand side of (3.9). It is easy to see that

$$\begin{aligned} \Big(\sum_{i=1}^m \alpha_i \nabla g_i(\bar y)\Big)_{(x_k,u_k,a_k)} &= \Big(\sum_{i=1}^m \alpha_i x_i^*, -\sum_{i=1}^m \alpha_i x_i^*, 0\Big), \\ \Big(\sum_{j=j_\tau(k)}^{j^\tau(k)} \xi_j \nabla d_j(\bar y)\Big)_{u_j} &= 2\xi_j \bar u_j \ \text{ for } \ j = 0, \ldots, k \ \text{ with } \ \xi_j = 0 \\ &\quad \text{if } \ j \notin \{j_\tau(k), \ldots, j^\tau(k)\}; \\ \Big(\sum_{j=0}^{k-1} (\nabla f_j(\bar y))^* p_{j+1}\Big)_{(x_j,u_j,a_j)} &= \begin{cases} -p_1 & \text{if } \ j = 0, \\ p_j - p_{j+1} & \text{if } \ j = 1, \ldots, k-1, \\ p_k & \text{if } \ j = k, \end{cases} \\ \Big(\sum_{j=0}^{k-1} (\nabla f_j(\bar y))^* p_{j+1}\Big)_{(X,U,A)} &= -h_k p = \big(-h_k p_1^x, \ldots, -h_k p_k^x, -h_k p_1^u, \ldots, -h_k p_k^u, \\ &\quad -h_k p_1^a, \ldots, -h_k p_k^a\big). \end{aligned}$$

Furthermore, the subdifferential sum rule from [36, Theorem 2.13] gives us the inclusion

$$\begin{aligned} \partial\varphi_0(\bar y) \subset \partial\varphi(\bar x_k) + \chi_0^k, \chi_1^k, 0, \ldots, 0) + h_k \sum_{j=0}^{k-1} \partial\ell(\bar x_j, \bar u_j, \bar a_j, \bar X_j, \bar U_j, \bar A_j) \\ + \sum_{j=0}^{k-1} \nabla\rho_j(\bar y) + \partial\sigma(\bar y), \end{aligned}$$

where χ_0^k, χ_1^k are defined in (2.60), where the functions $\rho(\cdot)$ and $\sigma(\cdot)$ are defined by

$$\rho_j(y) := \int_j^{j+1} \left\|(X_j, U_j, A_j) - \dot{\bar{z}}(t)\right\|^2 dt \quad \text{and}$$

$$\sigma(y) := \text{dist}^2\Big(\Big\|\frac{u_1^k - u_0^k}{h_k}\Big\|; \big(-\infty, \widetilde{\mu}\big]\Big) + \text{dist}^2\Big(\sum_{j=0}^{k-2} \|U_{j+1} - U_j\|; \big(-\infty, \widetilde{\mu}\big]\Big).$$

Note that the function $\psi(x) := \text{dist}^2(x; (-\infty, \widetilde{\mu}])$ is obviously differentiable with $\nabla\phi(x) = 0$ for all $x \le \widetilde{\mu}$. Combining this with second estimate in (2.40) yields $\partial\sigma(\bar{y}) = \{0\}$. Observe also that the nonzero part of $\nabla\rho_j(\bar{y})$ is calculated by $\nabla_{X_j,U_j,A_j}\rho(\bar{y}) = (\theta_j^x, \theta_j^u, \theta_j^a)$, where the latter triple is defined in (2.61). Hence the set $\lambda\partial\varphi_0(\bar{y})$ in (3.9) is represented as the collection of

$$\lambda(h_k w_0^x + \chi_0^k, h_k w_1^x + \chi_1^k, \ldots, h_k w_{k-1}^x, \vartheta^k, h_k w_0^u, \ldots, h_k w_{k-1}^u, 0, h_k w_0^a, \ldots, h_k w_{k-1}^a, 0,$$
$$\theta_0^x + h_k v_0^x, \ldots, \theta_{k-1}^x + h_k v_{k-1}^x, \theta_0^u + h_k v_0^u, \ldots, \theta_{k-1}^u + h_k v_{k-1}^u, \theta_0^a + h_k v_0^a, \ldots,$$
$$\theta_{k-1}^a + h_k v_{k-1}^a)$$

where $\vartheta^k \in \partial\varphi(\bar{x}_k)$, χ_j^k is defined in (2.60), and the components of $(w^x, w^u, w^a, v^x, v^u, v^a)$ satisfy (2.59). This together with (2.66) and the above gradient formulas shows that (3.9) amounts to the relationships

$$-x_{0k}^* - x_{00}^* = \lambda h_k w_0^x + \chi_0^k - p_1^x, \tag{2.67}$$

$$-x_{jj}^* = \lambda h_k w_j^x + \chi_j^k + p_j^x - p_{j+1}^x \quad \text{for} \quad j = 1, \ldots, k-1 \quad \text{with } \chi_j = 0 \;\; \text{if} \;\; j \neq 1, \tag{2.68}$$

$$0 = \lambda\vartheta^k + \sum_{i=1}^m \alpha_i x_i^* + p_k^x, \tag{2.69}$$

$$-u_{0k}^* - u_{00}^* = \lambda h_k w_0^u + 2\xi_0\bar{u}_0 - p_1^u, \tag{2.70}$$

$$-u_{jj}^* = \lambda h_k w_j^u + 2\xi_j\bar{u}_j + p_j^u - p_{j+1}^u, \tag{2.71}$$

$$0 = -\sum_{i=1}^{k} \alpha_i x_i^* + p_k^u + 2\xi_k \bar{u}_k, \tag{2.72}$$

$$-a_{0k}^* - a_{00}^* = \lambda h_k w_0^a - p_1^a, \tag{2.73}$$

$$-a_{jj}^* = \lambda h_k w_j^a + p_j^a - p_{j+1}^a \ \text{ for } \ j = 1, \ldots, k-1, \tag{2.74}$$

$$0 = p_k^a, \tag{2.75}$$

$$-X_{jj}^* = \lambda(h_k v_j^x + \theta_j^x) - h_k p_{j+1}^x \ \text{ for } \ j = 0, \ldots, k-1, \tag{2.76}$$

$$0 = \lambda(h_k v_j^u + \theta_j^u) - h_k p_{j+1}^u \ \text{ for } \ j = 0, \ldots, k-1, \tag{2.77}$$

$$0 = \lambda(h_k v_j^a + \theta_j^a) - h_k p_{j+1}^a \ \text{ for } \ j = 0, \ldots, k-1. \tag{2.78}$$

Now we are ready to justify all the conditions claimed in the theorem. Observe first that (2.64) clearly yields (2.54). Next we extend the vector p by a zero component by putting $p_0 := (x_{0k}^*, u_{0k}^*, a_{0k}^*)$. Then the conditions in (2.56) follow from (3.20), (3.23), and (3.26). Furthermore, the conditions in (3.4) follow from (3.28) and (3.29). Using the relationships

$$\frac{p_{j+1}^x - p_j^x}{h_k} - \lambda w_j^x - \frac{\chi_j^k}{h_k} = \frac{x_{jj}^*}{h_k}, \quad \frac{p_{j+1}^u - p_j^u}{h_k} - \lambda w_j^u = \frac{u_{jj}^*}{h_k} + 2\frac{\xi_j}{h_k}\bar{u}_j,$$

$$\frac{p_{j+1}^a - p_j^a}{h_k} - \lambda w_j^a = \frac{a_{jj}^*}{h_k}, \quad p_{j+1}^x - \frac{1}{h_k}\lambda\theta_{xj} - \lambda v_j^{xk} = \frac{X_{jj}^*}{h_k},$$

which hold due to (3.19), (3.22), (3.25), and (3.27), and then substituting them into the left-hand side of (3.13), we arrive at (3.5) for all $j = 0, \ldots, k-1$. To verify the nontriviality condition (2.53), suppose by contradiction that $\lambda = 0$, $\alpha = 0$, $\xi = 0$, $p_0^{uk} = 0$, $p_0^{ak} = 0$, and $p_j^x = 0$ for $j = 0, \ldots, k-1$. Then (3.20) yields that $p_k^x = 0$ and thus $p_j^x = 0$ for all $j = 0, \ldots, k$.

Observe further that $x^*_{0k} = p^x_0 = 0$, and so the conditions in (3.18), (3.19), and (3.27) imply that $x^*_{jj} = 0$ and $X^*_{jj} = 0$ for $j = 0, \ldots, k-1$. The validity of (3.28) and (3.29) ensures that $p^u_j = 0$ and $p^a_j = 0$ for $j = 1, \ldots, k$, which in turn show by (3.21), (3.22), (3.24), and (3.25) that $u^*_{jj} = 0$ and $a^*_{jj} = 0$ for $j = 0, \ldots, k-1$. As mentioned above, all the components of y^*_j different from $(x^*_{jj}, u^*_{jj}, a^*_{jj}, X^*_{jj})$ are zero for $j = 0, \ldots, k-1$. Hence we have $y^*_j = 0$ for $j = 0, \ldots, k-1$ and similarly $y^*_k = 0$ since the only potential nonzero component of this vector is $x^*_{0k} = p^x_0 = 0$. We get therefore that $y^*_j = 0$ for all $j = 0, \ldots, k$, which violates the nontriviality condition for (MP) and thus completes the proof of the theorem. □

The final result of this section employs the effective coderivative calculations for the sweeping control mapping taken from Theorem (2.6) that allows us to obtain necessary optimality conditions in (P^τ_k) expressed entirely via the given problem data and the minimizer under consideration under the additional assumption on the smoothness of f, which is imposed for simplicity. Furthermore, we derive an enhanced nontriviality relation in the case of linear independence of the generating vectors x^*_i for the underlying convex polyhedron C from (1.7).

Theorem 2.8 (optimality conditions for discretized sweeping inclusions via their initial data). *Let $\bar{z}^k = (\bar{x}^k, \bar{u}^k, \bar{a}^k)$ be an optimal solution to problem (P^τ_k) in the general framework of Theorem (2.7) with F given by (2.7) via the active constraint indices $I(\cdot)$ in (2.6) and locally smooth perturbation function f, let the active index subsets $I_0(\cdot)$ and $I_>(\cdot)$ be taken from (2.49), and let the triples $(\theta^{xk}_j, \theta^{uk}_j, \theta^{ak}_j)$ be defined in (2.61). Then there exist dual elements (λ^k, ξ^k, p^k) as in Theorem (2.7) together with vectors $\eta^k_j \in \mathbb{R}^m_+$ as $j = 0, \ldots, k$*

and $\gamma_j^k \in \mathbb{R}^m$ *as* $j = 0, \ldots, k-1$ *satisfying* (2.55)*, the* NONTRIVIALITY CONDITION

$$\lambda^k + \|\eta_k^k| + \|\xi^k\| + \sum_{j=0}^{k-1} \|p_j^{xk}\| + \|p_0^{uk}\| + \|p_0^{ak}\| \neq 0, \tag{2.79}$$

the PRIMAL-DUAL DYNAMIC EQUATIONS *for all* $j = 0, \ldots, k-1$ *with* χ_j^k *defined in* (2.60)*:*

$$\frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{-h_k} - f(\bar{x}_j^k, \bar{a}_j^k) = \sum_{i \in I(\bar{x}_j^k - \bar{u}_j^k)} \eta_{ji}^k x_i^*, \tag{2.80}$$

$$\begin{aligned} &\frac{p_{j+1}^{xk} - p_j^{xk}}{h_k} - \lambda^k w_j^{xk} - \frac{\chi_j^k}{h_k} = \nabla_x f(\bar{x}_j^k, \bar{a}_j^k)^* \left(\lambda^k (v_j^{xk} + h_k^{-1}\theta_j^{xk}) - p_{j+1}^{xk}\right) \\ &+ \sum_{i \in I_0\left(-p_{j+1}^{xk} + \lambda^k (h_k^{-1}\theta_j^{xk} + v_j^{xk})\right) \cup I_>\left(-p_{j+1}^{xk} + \lambda^k (h_k^{-1}\theta_j^{xk} + v_j^{xk})\right)} \gamma_{ji}^k x_i^*, \end{aligned} \tag{2.81}$$

$$\begin{aligned} &\frac{p_{j+1}^{uk} - p_j^{uk}}{h_k} - \lambda^k w_j^{uk} - \frac{2}{h_k}\xi_j^k \bar{u}_j^k \\ &= - \sum_{i \in I_0\left(-p_{j+1}^{xk} + \lambda^k (h_k^{-1}\theta_j^{xk} + v_j^{xk})\right) \cup I_>\left(-p_{j+1}^{xk} + \lambda^k (h_k^{-1}\theta_j^{xk} + v_j^{xk})\right)} \gamma_{ji}^k x_i^*, \end{aligned} \tag{2.82}$$

$$\frac{p_{j+1}^{ak} - p_j^{ak}}{h_k} - \lambda^k w_j^{ak} = \nabla_a f(\bar{x}_j^k, \bar{a}_j^k)^* \left(\lambda^k (v_j^{xk} + h_k^{-1}\theta_j^{xk}) - p_{j+1}^{xk}\right) \tag{2.83}$$

with $(w_j^{xk}, w_j^{uk}, w_j^{ak}, v_j^{xk}, v_j^{uk}, v_j^{ak})$ *taken from* (2.59)*, and the right endpoint* TRANSVERSALITY CONDITIONS

$$-p_k^{xk} \in \lambda^k \partial\varphi(\bar{x}_k^k) + \sum_{i=1}^m \eta_{ki}^k x_i^*, \quad p_k^{uk} = \sum_{i=1}^m \eta_{ki}^k x_i^* - 2\xi_k^k \bar{u}_k^k, \quad p_k^{ak} = 0 \tag{2.84}$$

such that the following implications hold:

$$\left[\langle x_i^*, \bar{x}_j^k - \bar{u}_j^k \rangle < 0\right] \Longrightarrow \eta_{ji}^k = 0, \tag{2.85}$$

$$\begin{cases} i \in I_0\left(-p^{xk}_{j+1} + \lambda^k(h_k^{-1}\theta^{xk}_j + v^{xk}_j)\right) \Longrightarrow \gamma^k_{ji} \in \mathbb{R}, \\ i \in I_>\left(-p^{xk}_{j+1} + \lambda^k(h_k^{-1}\theta^{xk}_j + v^{xk}_j)\right) \Longrightarrow \gamma^k_{ji} \geq 0, \\ \left[i \notin I_0\left(-p^{xk}_{j+1} + \lambda^k(h_k^{-1}\theta^{xk}_j + v^{xk}_j)\right) \cup I_>\left(-p^{xk}_{j+1} + \lambda^k(h_k^{-1}\theta^{xk}_j + v^{xk}_j)\right)\right] \\ \Longrightarrow \gamma^k_{ji} = 0 \end{cases} \tag{2.86}$$

for $j = 0, \ldots, k-1$ *and* $i = 1, \ldots, m$. *Furthermore, we have the constraint conditions* (2.55) *together with*

$$\left[\langle x^*_i, \bar{x}^k_j - \bar{u}^k_j\rangle < 0\right] \Longrightarrow \gamma^k_{ji} = 0 \;\; \textit{for} \;\; j = 0, \ldots, k-1 \;\; \textit{and} \;\; i = 1, \ldots, m, \tag{2.87}$$

$$\left[\langle x^*_i, \bar{x}^k_k - \bar{u}^k_k\rangle < 0\right] \Longrightarrow \eta^k_{ki} = 0 \;\; \textit{for} \;\; i = 1, \ldots, m. \tag{2.88}$$

Finally, the linear independence of the vectors $\{x^*_i|\, i \in I(\bar{x}^k_j - \bar{u}^k_j)\}$ *ensures the implication*

$$\eta^k_{ji} > 0 \Longrightarrow \left[\left\langle x^*_i, -p^{xk}_{j+1} + \lambda^k\left(h_k^{-1}\theta^{xk}_j + v^{xk}_j\right)\right\rangle = 0\right] \tag{2.89}$$

and the validity of the ENHANCED NONTRIVIALITY CONDITION

$$\lambda^k + \|\xi^k\| + \|p^{uk}_0\| + \|p^{xk}_1\| \neq 0. \tag{2.90}$$

Proof. The coderivative definition (2.48) allows us to equivalently rewrite the discrete Euler-Lagrange inclusion (3.5) in the coderivative form

$$\begin{aligned} &\left(\frac{p^{xk}_{j+1} - p^{xk}_j}{h_k} - \lambda^k w^{xk}_j - \frac{\chi^k_j}{h_k}, \frac{p^{uk}_{j+1} - p^{uk}_j}{h_k} - \lambda^k w^{uk}_j - \frac{2}{h_k}\xi^k_j \bar{u}^k_j, \frac{p^{ak}_{j+1} - p^{ak}_j}{h_k} - \lambda^k w^{ak}_j\right) \\ &\in D^*F\left(\bar{x}^k_j, \bar{u}^k_j, \bar{a}^k_j, \frac{\bar{x}^k_{j+1} - \bar{x}^k_j}{-h_k}\right)\left(\lambda^k(h_k^{-1}\theta^{xk}_j + v^{xk}_j) - p^{xk}_{j+1}\right), \quad j = 0, \ldots, k-1. \end{aligned} \tag{2.91}$$

Taking into account that $\dfrac{\bar{x}^k_{j+1} - \bar{x}^k_j}{-h_k} - f\left(\bar{x}^k_j, \bar{a}^k_j\right) \in G\left(\bar{x}^k_j - \bar{u}^k_j\right)$ for $j = 0, \ldots, k-1$ with $G(x) = N(x; C)$, we find by (2.91) vectors $\eta^k_j \in \mathbb{R}^m_+$ for $j = 0, \ldots, k-1$ such that conditions (2.80) and (3.30) hold. Using now the coderivative inclusion (2.50) from Theorem (2.6) with

$x := \bar{x}_j^k,\ u := \bar{u}_j^k,\ a := \bar{a}_j^k,\ w := \dfrac{\bar{x}_{j+1}^k - \bar{x}_j^k}{-h_k}$, and $y := \lambda^k(h_k^{-1}\theta_j^{xk} + v_j^{xk}) - p_{j+1}^{xk}$ shows $\gamma_j^k \in \mathbb{R}^m$ and the relationships

$$\Big(\frac{p_{j+1}^{xk} - p_j^{xk}}{h_k} - \lambda^k w_j^{xk} - \frac{\chi_j^k}{h_k}, \frac{p_{j+1}^{uk} - p_j^{uk}}{h_k} - \lambda^k w_j^{uk} - \frac{2}{h_k}\xi_j^k \bar{u}_j^k, \frac{p_{j+1}^{ak} - p_j^{ak}}{h_k} - \lambda^k w_j^{ak}\Big)$$
$$= \Bigg(\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)^* \left(\lambda^k(v_j^{xk} + h_k^{-1}\theta_j^{xk}) - p_{j+1}^{xk}\right)$$
$$+ \sum_{i \in I_0\left(-p_{j+1}^{xk} + \lambda^k(h_k^{-1}\theta_j^{xk} + v_j^{xk})\right) \cup I_>\left(-p_{j+1}^{xk} + \lambda^k(h_k^{-1}\theta_j^{xk} + v_j^{xk})\right)} \gamma_{ji}^k x_i^*,$$
$$- \sum_{i \in I_0\left(-p_{j+1}^{xk} + \lambda^k(h_k^{-1}\theta_j^{xk} + v_j^{xk})\right) \cup I_>\left(-p_{j+1}^{xk} + \lambda^k(h_k^{-1}\theta_j^{xk} + v_j^{xk})\right)} \gamma_{ji}^k x_i^*, \nabla_a f(\bar{x}_j^k, \bar{a}_j^k)$$
$$^* \left(\lambda^k(v_j^{xk} + h_k^{-1}\theta_j^{xk}) - p_{j+1}^{xk}\right)\Bigg)$$

are satisfied for all $j = 0, \ldots, k-1$ and thus ensure the validity of all the conditions in (2.81), (2.82), (2.83), (2.86), and (2.87). Defining now $\eta_k^k := \alpha_k$ via α_k from the statement of Theorem (2.53) yields $\eta_j^k \in \mathbb{R}_+^m$ for $j = 0, \ldots, k$ and allows us to deduce the nontriviality condition (2.79) from that in (2.53) and the transversality conditions in (2.84) from those in (2.56) and (3.4). Implication (2.88) is a direct consequence of (2.54) and the definition of η_k^k.

Assume finally that the generating vectors $\{x_i^* |\ i \in I(\bar{x}_j^k - \bar{u}_j^k)\}$ of the convex polyhedron C are linearly independent. It is not hard to observe that the inclusion

$$\lambda^k(h_k^{-1}\theta_j^{xk} + v_j^{xk}) - p_{j+1}^{xk} \in \operatorname{dom} D^*G\left(\bar{x}_j^k - \bar{u}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{-h_k} - f(\bar{x}_j^k, \bar{a}_j^k)\right),$$

which follows from (2.91), in this case yields (2.89) due to (2.51). It remains to verify the enhanced nontriviality condition (2.90) under the imposed linear independence. Suppose on the contrary that $\lambda^k = 0$, $\xi^k = 0$, $p_0^{uk} = 0$, and $p_1^{xk} = 0$. Then $p_j^{uk} = 0$ for $j = 0, \ldots, k$ and $p_j^{ak} = 0$ for $j = 1, \ldots, k$ by (3.4). Furthermore, it follows from the second condition in (2.84) with $p_k^{uk} = 0$ that $\sum_{i=1}^{m} \eta_{ki}^k x_i^* = 0$. This implies by definition (2.6) of the active constraint indices and the imposed linear independence of x_i^* over this index set that $\eta_k^k = 0$, and so $p_k^{xk} = 0$ by the first condition in (2.84). On the other hand, we get from (2.82) that

$$\sum_{i \in I_0\left(-p_{j+1}^{xk}+\lambda^k(h_k^{-1}\theta_j^{xk}+v_j^{xk})\right) \cup I_>\left(-p_{j+1}^{xk}+\lambda^k(h_k^{-1}\theta_j^{xk}+v_j^{xk})\right)} \gamma_{ji}^k x_i^* = 0.$$

Combining this with (2.81) and $p_k^{xk} = 0$ shows that $p_j^{xk} = 0$ for all $j = 2, \ldots, k-1$. It then follows from (2.81) that

$$\begin{cases} p_2^{xk} - p_1^{xk} = \chi_1^k + h_k \nabla_x f(\bar{x}_1^k, \bar{a}_1^k)(-p_2^{xk}), \\ p_1^{xk} - p_0^{xk} = \chi_0^k + h_k \nabla_x f(\bar{x}_0^k, \bar{a}_0^k)(-p_1^{xk}). \end{cases}$$

Since $p_2^{xk} = 0$ and $p_1^{xk} = 0$ then $\chi_1^k = 0$ and thus $\chi_0^k = -\chi_1^k = 0$, which implies $p_0^{xk} = 0$. We finally deduce from (2.83) that $p_0^{ak} = 0$. This clearly implies that (2.79) is violated and hence justifies the validity of (2.90). □

CHAPTER 3 OPTIMALITY CONDITIONS FOR A CONTROLLED SWEEPING PROCESS

In this chapter we derive necessary optimality conditions for relaxed intermediate local minimizers of the sweeping control problem (P^τ) under consideration in the general case of $0 \le \tau \le \overline{\tau} = \min\{r, T\}$ with some specifications and improvements in the case where τ is not an endpoint.

Proposition 3.1 (precise calculating the coderivatives of the normal cone mappings associated with convex polyhedra). *Let $G(x) = N(x;C)$ be the normal cone mapping associated with the convex polyhedron* (1.7)*, and let the featured active index subsets $I_0(\cdot)$ and $I_>(\cdot)$ be defined in* (2.49)*. Given $(\bar{x},\bar{y}) \in \operatorname{gph} G$, assume that the generating elements $\{x_i^*|\ i \in I(\bar{x})\}$ of* (1.4) *along the active constraint indices* (2.6) *are linearly independent. Then we have the coderivative expression*

$$D^*G(\bar{x},\bar{y})(u) = \operatorname{span}\big\{x_i^*\big|\ i \in I_0(u)\big\} + \operatorname{cone}\big\{x_i^*\big|\ i \in I_>(u)\big\} \ \ \textit{for all}\ \ u \in \operatorname{dom} D^*G(\bar{x},\bar{y}),$$

where the latter coderivative domain is characterized by

$$u \in \operatorname{dom} D^*G(\bar{x},\bar{y}) \Longleftrightarrow \big[i \in J(\bar{x},\bar{y}) \Longrightarrow \langle x_i^*, u\rangle = 0\big]$$

via the so-called strict complementarity subset of active indices $J(\bar{x},\bar{y}) := \{i \in I(\bar{x})|\ \lambda_i > 0\}$ for a unique collection of the multipliers $\lambda_i \ge 0$ coming from the representation $\bar{y} = \sum_{i\in I(\bar{x})} \lambda_i x_i^*$.

Before establishing the aforementioned necessary optimality conditions for (P^τ) we make the following remark. It is well known that the subdifferential mapping used in condition (2.59) of Theorem 2.7 is *robust* (i.e., closed-graph) with respect to subdifferentiation variables. However, in the discrete-time and continuous-time settings under consideration allows the dependence of the running cost ℓ on the time parameter, which is not under subdifferentiation. For the limiting procedure in what follows, we *require* the subdifferential robustness with respect to the time parameter. It is not a restrictive assumption that holds, in particular, for smooth functions with time-continuous derivatives as well as in rather general nonsmooth settings discussed, e.g., in [35, 37]. We recall also that (as assumed in [10, Theorem 3.1]) that the local optimal solution $\bar{z}(\cdot)$ under consideration satisfies the sweeping differential inclusion (1.9) at all the the mesh points with the right and left derivatives at $t = 0$ and $t = T$, respectively.

Theorem 3.2 (general necessary optimality conditions for the perturbed sweeping process). *Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot))$ be a r.i.l.m. for problem (P^τ) with any $\tau \in [0, \overline{\tau}]$. In addition to the assumptions of Theorem 2.8, suppose that ℓ in (1.5) is continuous in t a.e. on $[0, T]$ and admits the representation*

$$\ell(t, z, \dot{z}) = \ell_1(t, z, \dot{x}) + \ell_2(t, \dot{u}) + \ell_3(t, \dot{a}), \tag{3.1}$$

where the (local) Lipschitz constants of $\ell_1(t, \cdot, \cdot)$ and $\ell_3(t, \cdot)$ can be chosen as essentially bounded on $[0, T]$ and continuous at a.e. $t \in [0, T]$ including $t = 0$, while ℓ_2 is differentiable in $\dot{u}$ on $\mathbb{R}^n$ satisfying

$$\|\nabla_{\dot{u}} \ell_2(t, \dot{u}, \dot{a})\| \le L\|\dot{u}\| \ \text{ and } \ \|\nabla_{\dot{u}} \ell_2(t, \dot{u}_1) - \nabla_{\dot{u}} \ell_2(t, \dot{u}_2)\| \le L|t - s| + L\|\dot{u}_1 - \dot{u}_2\| \tag{3.2}$$

for some constant $L > 0$, all numbers $t, s \in [0, T]$, $\dot{a} \in \mathbb{R}^d$, and all vectors $\dot{u}, \dot{u}_1, \dot{u}_2 \in \mathbb{R}^n$.

Then there exist a number $\lambda \ge 0$, an adjoint arc $p(\cdot) = (p^x(\cdot), p^u(\cdot), p^a(\cdot)) \in W^{1,2}([0, T];$ $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d)$, subgradient functions $w(\cdot) = (w^x(\cdot), w^u(\cdot), w^a(\cdot)) \in L^2([0, T]; \mathbb{R}^{2n+d})$ and

$v(\cdot) = (v^x(\cdot, v^u(\cdot), v^a(\cdot)) \in L^2([0,T];\mathbb{R}^{2n+d})$ *well defined at* $t = 0$ *and satisfying the inclusion*

$$\big(w(t), v(t)\big) \in \mathrm{co}\,\partial\ell\big(t, \bar{z}(t), \dot{\bar{z}}(t)\big) \;\; \text{for a.e. } \; t \in [0,T], \tag{3.3}$$

and measures $\gamma = (\gamma_1, \ldots, \gamma_n) \in C^*([0,T];\mathbb{R}^n)$, $\xi \in C^*([0,T];\mathbb{R})$ *on* $[0,T]$ *such that we have:*

- Primal-dual dynamic relationships:

$$-\dot{\bar{x}}(t) = \sum_{i=1}^{m} \eta_i(t) x_i^* + f\big(\bar{x}(t), \bar{a}(t)\big) \;\; \text{for a.e. } \; t \in [0,T], \tag{3.4}$$

where $\eta_i(\cdot) \in L^2([0,T];\mathbb{R}_+)$ *are well defined at* $t = T$ *being uniquely determined by representation* (3.4)*;*

$$\dot{p}(t) = \lambda w(t) + \Big(\nabla_x f\big(\bar{x}(t), \bar{a}(t)\big)^*\big(\lambda v^x(t) - q^x(t)\big), 0, \nabla_a f\big(\bar{x}(t), \bar{a}(t)\big)^*\big(\lambda v^x(t) - q^x(t)\big)\Big), \tag{3.5}$$

$$q^u(t) = \lambda \nabla_{\dot{u}} \ell\big(t, \dot{\bar{u}}(t)\big), \quad q^a(t) \in \lambda \partial_{\dot{a}} \ell_3\big(t, \dot{\bar{a}}(t)\big) \;\; \text{for a.e. } \; t \in [0,T], \tag{3.6}$$

where the vector function $q = (q^x, q^u, q^a) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ *is of bounded variation with its left-continuous representative given for all* $t \in [0,T]$*, except at most a countable subset, by*

$$q(t) := p(t) - \int_{[t,T]} \big(d\gamma(s), 2\bar{u}(s) d\xi(s) - d\gamma(s), 0\big). \tag{3.7}$$

Moreover, for a.e. $t \in [0,T]$ *including* $t = T$ *and all* $i = 1, \ldots, m$ *we have the implications*

$$\langle x_i^*, \bar{x}(t) - \bar{u}(t)\rangle < 0 \Longrightarrow \eta_i(t) = 0, \quad \eta_i(t) > 0 \Longrightarrow \langle x_i^*, \lambda v^x(t) - q^x(t)\rangle = 0. \tag{3.8}$$

- Transversality conditions *at the right and left endpoints, respectively:*

$$\begin{cases} -p^x(T) - \displaystyle\sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) x_i^* \in \lambda \partial\varphi\big(\bar{x}(T)\big), \\ p^u(T) - \displaystyle\sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) x_i^* \in 2\bar{u}(T) N\big(\|\bar{u}(T)\|; [r - \tau, r + \tau]\big), \quad p^a(T) = 0; \end{cases} \tag{3.9}$$

$$\begin{cases} q^x(0) \in \mathbb{R}^n, \quad q^a(0) = \lambda v^a(0), \quad q^u(0) \in \lambda v^u(0) - 2\bar{u}(0)N\big(\|\bar{u}(0)\|; [r-\tau, r+\tau]\big) \\ \\ +D^*G\big(x_0 - \bar{u}(0), -\dot{\bar{x}}(0) - f(\bar{x}(0), \bar{a}(0))\big)\big(-q^x(0) + \lambda v^x(0)\big) \end{cases} \tag{3.10}$$

with the coderivative D^*G of $G(\cdot) = N(\cdot; C)$ explicitly calculated in Proposition 3.1 *and with*

$$\sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) x_i^* \in N\big(\bar{x}(T) - \bar{u}(T); C\big). \tag{3.11}$$

• Measure nonatomicity conditions:

(a) *If $t \in [0, T)$ and $\langle x_i^*, \bar{x}(t) - \bar{u}(t)\rangle < 0$ for all $i = 1, \ldots, m$, then there exists a neighborhood V_t of t in $[0, T)$ such that $\gamma(V) = 0$ for all Borel subsets V of V_t.*

(b) *Assume that $\tau \in (0, \overline{\tau})$ and take any $t \in [0, \tau) \cup (T - \tau, T]$ with $r - \tau < \|\bar{u}(t)\| < r + \tau$. Then there exists a neighborhood W_t of t in $(0, \tau) \cup (T - \tau, T)$ such that $\xi(W) = 0$ for all Borel subsets W of W_t.*

• Nontriviality conditions:

(a) *Impose one of the following assumptions on the local minimizer $\bar{z}(\cdot)$ and the data of (P^τ) as $\tau \in [0, \overline{\tau}]$:*

$$\text{either} \;\; l(r + 2\tau) < (r - 2\tau)^2, \quad \text{or} \quad \langle \bar{x}(t), \bar{u}(t)\rangle \neq \|\bar{u}(t)\|^2 \;\; \text{for all} \;\; t \in [0, T), \tag{3.12}$$

where the constant $l > 0$ is calculated in (2.3) *with $u(\cdot) = \bar{u}(\cdot)$.*[1] *Then we have*

$$\lambda + \|q^x(0)\| + \|q^u(0)\| + \|p(T)\| > 0 \tag{3.13}$$

[1]Note that the first condition in (3.12) implies the second one for $\tau = 0$, while in general they are independent.

provided that either $\tau < r$ *or* $\bar{u}(T) \neq 0$.

(b) *If in addition* $0 < \tau < r$ *and the Jacobian* $\nabla_a f(x,a)$ *is surjective, the we have the following enhanced nontriviality conditions while imposing the corresponding endpoint interiority assumptions:*

$$\big[\langle x_i^*, x_0 - \bar{u}(0)\rangle < 0,\ i = 1,\ldots,m\big] \Longrightarrow \big[\lambda + \|q^u(0)\| + \|p(T)\| > 0\big], \tag{3.14}$$

$$\big[\langle x_i^*, x_0 - \bar{u}(0)\rangle < 0,\ r-\tau < \|\bar{u}(0)\| < r+\tau,\ i = 1,\ldots,m\big] \Longrightarrow \big[\lambda + \|p(T)\| > 0\big], \tag{3.15}$$

$$\big[\langle x_i^*, \bar{x}(T) - \bar{u}(T)\rangle < 0,\ r-\tau < \|\bar{u}(T)\| < r+\tau,\ i = 1,\ldots,m\big] \Longrightarrow \big[\lambda + \|q^x(0)\| + \|q^u(0)\| > 0\big]. \tag{3.16}$$

Proof. The derivation of the necessary optimality conditions for the given r.i.l.m. $\bar{z}(\cdot)$ in problem (P^τ) is based on passing the limit as $k \to \infty$ from the optimality conditions for the strongly convergent sequence $\bar{z}^k(\cdot) \to \bar{z}(\cdot)$ of optimal solutions to the discrete problems (P_k^τ) obtained in Theorem 2.8. The proof is rather involved, and for the reader's convenience we split it into several steps.

Step 1: *Subdifferential inclusion.* Let us first justify (3.3). For each $k \in I\!N$ define the functions $w^k, v^k \colon [0,T] \to \mathbb{R}^{2n+d}$ as piecewise constant extensions to $[0,T]$ of the vectors w_j^k and v_j^k that are defined on the mesh Δ_k and satisfy the subdifferential inclusion (2.59) therein. The assumptions made and the structure of ℓ in (3.1), (3.2) ensure that the subgradient sets $\partial\ell(t,\cdot)$ are uniformly bounded near $\bar{z}(\cdot)$ by the L^2-Lipschitz constant of ℓ, and thus the sequence $\{(w^k(\cdot), v^k(\cdot))\}$ is weakly compact in $L^2([0,T];\mathbb{R}^{2(2n+d)}) := L^2[0,T]$. This allows us to select a subsequence (no relabeling hereafter) converging

$$\big(w^k(\cdot), v^k(\cdot)\big) \to \big(w(t), v(t)\big) \text{ weakly in } L^2[0,T] \text{ as } k \to \infty$$

for some $(w(\cdot), v(\cdot)) \in L^2[0,T]$. Furthermore, the local Lipschitz continuity of $\ell(0,\cdot,\cdot)$ yields by (2.59) for $j=0$ that the sequence $\{(w_0^k, v_0^k)\}$ is bounded and hence converges as $k \to \infty$ to a pair $(w_0, v_0) =: (w(0), v(0))$ along a subsequence. It follows from the aforementioned Mazur weak closure theorem that there are convex combinations of $(w^k(\cdot), v^k(\cdot))$, which converge to $(w(\cdot), v(\cdot))$ in the L^2-norm and hence a.e. on $[0,T]$ for some subsequence. Then passing to the limit in (2.59) along the latter subsequence and taking into account the assumed a.e. continuity of the running cost ℓ in t and robustness of its subdifferential in $(z,\dot z)$ with respect to all the variables, we arrive at the convexified inclusion (3.3).

Step 2: *Passing to the limit in the primal equation.* Our next aim is to arrive at the primal equation (3.4) and the first implication in (3.8) with the corresponding functions $\eta_i(\cdot)$ by passing to the limit in (2.80), (2.87) and (2.89). We start with considering the functions

$$\theta^k(t) := \frac{\theta_j^k}{h_k} \text{ as } t \in [t_j^k, t_{j+1}^k),\ j = 0,\ldots,k-1, \quad k \in I\!N,$$

on $[0,T]$ with θ_j^k taken from (2.61). It follows from the convergence $\bar z^k(\cdot) \to \bar z(\cdot)$ in Theorem 2.8 that

$$\begin{aligned}\int_0^T \|\theta^{xk}(t)\|^2 dt =& \sum_{j=0}^{k-1} \frac{\|\theta_j^{xk}\|^2}{h_k} \le \frac{4}{h_k}\sum_{j=0}^{k-1}\Big(\int_{t_j^k}^{t_{j+1}^k}\Big\|\dot{\bar x}(t) - \frac{\bar x_{j+1}^k - \bar x_j^k}{h_k}\Big\| dt\Big)^2 \\ \le& 4\sum_{j=0}^{k-1}\int_{t_j^k}^{t_{j+1}^k}\Big\|\dot{\bar x}(t) - \frac{\bar x_{j+1}^k - \bar x_j^k}{h_k}\Big\|^2 dt = 4\int_0^T \big\|\dot{\bar x}(t) - \dot{\bar x}^k(t)\big\|^2 dt \to 0\end{aligned} \tag{3.17}$$

and similarly for $\theta^{uk}(\cdot)$ and $\theta^{ak}(\cdot)$. This yields the a.e. convergence of these functions to zero on $[0,T]$. Moreover, the construction above shows that we can always have $\theta_0^k \to 0 =: \theta(0)$.

Further, it is easy to see that the assumed linear independence of $\{x_i^* \mid i \in I(\bar{x}(\cdot) - \bar{u}(\cdot))\}$ ensures the one for $\{x_i^* \mid i \in I(\bar{x}_j^k - \bar{u}_j^k)\}$ by definition (2.6) and the strong convergence of Theorem 2.8. This allows us to take the vectors $\eta_j^k \in \mathbb{R}_+^m$ from Theorem 2.8 and construct the piecewise constant functions $\eta^k(\cdot)$ on $[0,T]$ by $\eta^k(t) := \eta_j^k$ for $t \in [t_j^k, t_{j+1}^k)$ with $\eta^k(T) := \eta_k^k$. It follows from (2.86) that

$$-\dot{\bar{x}}^k(t) = \sum_{i=1}^m \eta_i^k(t) x_i^* + f\big(\bar{x}^k(t_j^k), \bar{a}^k(t_j^k)\big) \text{ whenever } t \in (t_j^k, t_{j+1}^k), \quad k \in I\!N, \tag{3.18}$$

via the corresponding components of $\eta^k(t)$. On the other hand, the feasibility of $\bar{z}(\cdot)$ to (P^τ) yields $-\dot{\bar{x}}(t) \in G(\bar{x}(t) - \bar{u}(t)) + f(\bar{x}(t), \bar{a}(t))$ for a.e. $t \in [0,T]$ with the closed-valued normal cone mapping $G(\cdot) = N(\cdot; C)$. Due to the measurability of $G(\cdot)$ by [45, Theorem 4.26] and the measurable selection result from [45, Corollary 4.6] we find nonnegative measurable functions $\eta_i(\cdot)$ on $[0,T]$ as $i = 1, \ldots, m$ such that equation (3.4) and the first implication in (3.8) hold. Combining (3.18) and (3.4) gives us

$$\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t) = \sum_{i=1}^m \big[\eta_i^k(t) - \eta_i(t)\big] x_i^* + f\big(\bar{x}^k(t_j^k), \bar{a}^k(t_j^k)\big) - f\big(\bar{x}(t), \bar{a}(t)\big)$$

for $t \in (t_j^k, t_{j+1}^k)$ and $j = 0, \ldots, k-1$. Thus we get the estimate

$$\left\| \sum_{i=1}^m \big[\eta_i(t) - \eta_i^k(t)\big] x_i^* \right\| \le \|\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t)\| + \big\| f\big(\bar{x}(t), \bar{a}(t)\big) - f\big(\bar{x}^k(t_j^k), \bar{a}^k(t_j^k)\big) \big\| \tag{3.19}$$

for $t \in (t_j^k, t_{j+1}^k)$. For each $k \in I\!N$ define now the function

$$\nu^k(t) := \max\big\{ t_j^k \big|\ t_j^k \le t,\ 0 \le j \le k \big\} \text{ for all } t \in [0,T]. \tag{3.20}$$

Passing to the limit in (3.19) with replacing t_j^k by $\nu(t)$ and taking into account the strong convergence $\bar{z}^k(\cdot) \to \bar{z}(\cdot)$ together with the continuity of f on the left-hand side of (3.19), we get

$$\sum_{i\in I(\bar{x}(t)-\bar{u}(t))}\big[\eta_i(t)-\eta_i^k(t)\big]x_i^*\to 0\ \text{ as }\ k\to\infty\ \text{ for a.e. }\ t\in[0,T].$$

Then the assumed linear independence of the generating vectors x_i^* with $i\in I(\bar{x}(t)-\bar{u}(t))$ ensures the a.e. convergence $\eta^k(t)\to\eta(t)$ on $[0,T]$ as $k\to\infty$. Furthermore, we will show that η_k^k converges to the well-defined vector $(\eta_1(T),\ldots,\eta_m(T))$ in Step 5. Proceeding similarly to the proof of [13, Theorem 6.1], we can justify the extra regularity $\eta(\cdot)\in L^2([0,T];\mathbb{R}^m_+)$, which however is not used in what follows.

Step 3: *Extensions of approximating dual elements.* Here we extend discrete dual elements from Theorem 2.8 to the whole interval $[0,T]$. First construct $q^k(t)=(q^{xk}(t),q^{uk}(t),\,q^{ak}(t))$ on $[0,T]$ as the piecewise linear extensions of $q^k(t_j^k):=p_j^k$ as $j=0,\ldots,k$. Then define $\gamma^k(t)$ on $[0,T]$ as piecewise constant $\gamma^k(t):=\gamma_j^k$ for $t\in[t_j^k,t_{j+1}^k)$ and $j=0,\ldots,k-1$ with $\gamma^k(t_k^k):=0$. We also set $\xi^k(t):=\dfrac{\xi_j^k}{h_k}$ for $t\in[t_j^k,t_{j+1}^k)$ and $j=0,\ldots,k-1$ with $\xi^k(t_k^k):=\xi_k^k$, and

$$\chi^k(t):=\begin{cases}\dfrac{\chi_j^k}{h_k} & \text{for } t\in[t_j^k,t_{j+1}^k) \text{ and } j=0,1,\\[2ex] 0 & \text{for } t\in[t_j^k,t_{j+1}^k) \text{ and } j=2,\ldots,k-1.\end{cases}$$

Using the function $\nu^k(t)$ given in (3.20), we deduce respectively from (2.81), (2.82), and (2.83) that

$$\begin{aligned}\dot{q}^{xk}(t)-\lambda^k w^{xk}(t)-\chi^k(t)=&\nabla_x f\big(\bar{x}^k(\nu^k(t)),\bar{a}^k(\nu^k(t))^*\big(\lambda^k(v^{xk}(t)+\theta^{xk}(t))-q^{xk}(\nu^k(t)+h_k)\big)\\ &+\sum_{i\in I_0(\lambda^k(v^{xk}(t)+\theta^{xk}(t))-q^{xk}(\nu^k(t)+h_k))\cup I_>(\lambda^k(v^{xk}(t)+\theta^{xk}(t))-q^{xk}(\nu^k(t)+h_k))}\gamma_i^k(t)x_i^*,\end{aligned}\tag{3.21}$$

$$\dot{q}^{uk}(t) - \lambda^k w^{uk}(t) = 2\xi^k(t)\bar{u}^k\big(\nu^k(t)\big) - \sum_{i\in I_0(\lambda^k(v^{xk}(t)+\theta^{xk}(t))-q^{xk}(\nu^k(t)+h_k))\cup I_>(\lambda^k(v^{xk}(t)+\theta^{xk}(t))-q^{xk}(\nu^k(t)+h_k))} \gamma_i^k(t)x_i^*, \tag{3.22}$$

$$\dot{q}^{ak}(t) - \lambda^k w^{ak}(t) = \nabla_a f\big(\bar{x}^k(\nu^k(t)),\bar{a}^k(\nu^k(t)\big)^*\big(\lambda^k(v^{xk}(t)+\theta^{xk}(t)) - q^{xk}(\nu^k(t)+h_k)\big) \tag{3.23}$$

for $t \in (t_j^k, t_{j+1}^k)$ and $j = 0,\ldots,k-1$. Next we define $p^k(t) = (p^{xk}(t), p^{uk}(t), p^{ak}(t))$ on $[0,T]$ by setting

$$p^k(t) := q^k(t) + \int_{[t,T]} \left(\sum_{i=1}^m \gamma_i^k(s)x_i^*, 2\xi^k(s)\bar{u}^k\big(\nu^k(s)\big) - \sum_{i=1}^m \gamma_i^k(s)x_i^*, 0\right) ds \tag{3.24}$$

for all $t \in [0,T]$. This gives us $p^k(T) = q^k(T)$ and the differential relation

$$\dot{p}^k(t) = \dot{q}^k(t) - \left(\sum_{i=1}^m \gamma_i^k(t)x_i^*, 2\xi^k(t)\bar{u}^k\big(\nu^k(t)\big) - \sum_{i=1}^m \gamma_i^k(t)x_i^*, 0\right) \quad \text{a.e. } t \in [0,T]. \tag{3.25}$$

It follows from (3.25), (3.21)–(3.23), and the definition of $I_0(\cdot)$ and $I_>(\cdot)$ in (2.49) that

$$\dot{p}^{xk}(t) - \lambda^k w^{xk}(t) - \chi^k(t) = \nabla_x f\big(\bar{x}^k(\nu^k(t)),\bar{a}^k(\nu^k(t)\big)^*\big(\lambda^k(v^{xk}(t)+\theta^{xk}(t)) - q^{xk}(\nu^k(t)+h_k)\big), \tag{3.26}$$

$$\dot{p}^{uk}(t) - \lambda^k w^{uk}(t) = 0, \tag{3.27}$$

$$\dot{p}^{ak}(t) - \lambda^k w^{ak}(t) = \nabla_a f\big(\bar{x}^k(\nu^k(t)),\bar{a}^k(\nu^k(t)\big)^*\big(\lambda^k(v^{xk}(t)+\theta^{xk}(t)) - q^{xk}(\nu^k(t)+h_k)\big), \tag{3.28}$$

for every $t \in (t_j^k, t_{j+1}^k)$, $j = 0,\ldots,k-1$. Define now the vector measures γ_{mes}^k and ξ_{mes}^k on $[0,T]$ by

$$\int_A d\gamma_{mes}^k := \int_A \sum_{i=1}^m \gamma_i^k(t)x_i^* dt, \quad \int_A d\xi_{mes}^k := \int_A \xi^k(t)dt \ \text{ for any Borel subset } A \subset [0,T] \tag{3.29}$$

with dropping further the symbol "*mes*" for simplicity. By taking into account the preservation of all the relationships in Theorem 2.8 by normalization and the above constructions of the extended functions on $[0,T]$, we can rewrite the nontriviality condition (2.54) as

$$\lambda^k+\|p^k(T)\|+\|q^{uk}(0)\|+\|q^{xk}(h_k)\|+\int_0^T|\xi^k(t)|dt+|\xi^k_k|+\int_0^T\left\|\sum_{i=1}^m\gamma_i^k(t)x_i^*\right\|dt=1,\quad k\in I\!N. \tag{3.30}$$

Step 4: *Passing to the limit in dual dynamic relationships.* Using (3.30) allows us to suppose without loss of generality that $\lambda^k\to\lambda$ as $k\to\infty$ for some $\lambda\geq 0$. Let us next verify that the sequence $\{(p_0^{xk},\ldots,p_k^{xk})\}_{k\in I\!N}$ is bounded in $\mathbb{R}^{(k+1)n}$. Indeed, we have by (2.81) that

$$p_j^{xk}=p_{j+1}^{xk}-\lambda^kh_kw_j^{xk}-\chi_j^k-\nabla_xf(\bar{x}_j^k,\bar{a}_j^k)^*(\lambda^kh_kv_j^{xk}+\lambda^k\theta_j^{xk}-h_kp_{j+1}^{xk})-h_k\sum_{i=1}^m\gamma_{ji}^kx_i^* \tag{3.31}$$

for $j=0,\ldots,k-1$. It follows from (3.17) and (3.30) that the quantities $\nabla_xf(\bar{x}_j^k,\bar{a}_j^k)$, $\lambda^k\theta_j^{xk}$, and $h_k\sum_{i=1}^m\gamma_{ji}^kx_i^*$ are uniformly bounded for $j=0,\ldots,k-1$ while $\chi_j^k\to 0$ as $k\to\infty$ due to definition (2.60) and the first condition in (2.27). Furthermore, the imposed structure (3.1) of ℓ and the assumptions on the Lipschitz constant $L(t)$ of the running cost in (1.5), which are equivalent to the Riemann integrability of $L(\cdot)$ on $[0,T]$, yield by (3.3) the relationships

$$\sum_{j=0}^{k-1}\|h_kw_j^{xk}\|=\sum_{j=0}^{k-1}\|h_kw^{xk}(t_j)\|\leq\sum_{j=0}^{k-1}h_kL(t_j)\leq 2\int_{[0,T]}L(t)dt=:\widetilde{L}<\infty \tag{3.32}$$

and ensure similarly that $\sum_{j=0}^{k-1}\|h_kv_j^{xk}\|<\widetilde{L}$.

We will justify the boundedness of $\{(p_0^{xk},\ldots,p_k^{xk})\}_{k\in I\!N}$. It follows from (3.31) that

$$\begin{aligned}\|p_j^{xk}\|\leq&\left(1+h_k\left\|\nabla_xf(\bar{x}_j^k,\bar{a}_j^k)\right\|\right)\left\|p_{j+1}^{xk}\right\|+\lambda^k\left(\left\|h_kw_j^{xk}\right\|+\left\|\nabla_xf(\bar{x}_j^k,\bar{a}_j^k)\right\|\left\|h_kv_j^{xk}\right\|\right)\\&+h_k\left\|\nabla_xf(\bar{x}_j^k,\bar{a}_j^k)\right\|\lambda^k\left\|\theta^{xk}(t_j)\right\|+h_k\left\|\sum_{i=1}^m\gamma_{ji}^kx_i^*\right\|+\|\chi_j^k\|,\end{aligned} \tag{3.33}$$

for all $j = 0, \ldots, k-1$. Denote by

$$A_j^k := \lambda^k \left(\left\|h_k w_j^{xk}\right\| + \left\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)\right\| \left\|h_k v_j^{xk}\right\|\right) + h_k \left\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)\right\| \lambda^k \left\|\theta^{xk}(t_j)\right\| + h_k \left\|\sum_{i=1}^m \gamma_{ji}^k x_i^*\right\|$$

for $j = 0, \ldots, k-1$. Let $M_1 > 0$ be a constant such that $\left\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)\right\| \leq M_1$ for all $j = 0, \ldots, k$ and $k \in I\!N$.

We have

$$h_k \left\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)\right\| \lambda^k \left\|\theta^{xk}(t_j)\right\| \leq M_1 h_k \left\|\theta^{xk}(t_j)\right\| = M_1 \sqrt{h_k \int_{t_j}^{t_{j+1}} \|\theta^{xk}(t)\|^2 dt}$$

for $j = 0, \ldots, k-1$, and hence

$$\sum_{j=0}^{k-1} h_k \left\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)\right\| \lambda^k \left\|\theta^{xk}(t_j)\right\| \leq M_1 \sqrt{\int_0^T \|\theta^{xk}(t)\|^2 dt} \downarrow 0 \;\text{ as }\; k \to \infty$$

On the other hand, we also have

$$\sum_{j=0}^{k-1} h_k \left\|\sum_{i=1}^m \gamma_{ji}^k x_i^*\right\| = \int_0^T \left\|\sum_{i=1}^m \gamma_i^k(t) x_i^*\right\| dt \leq 1$$

by (3.30).

We deduce from the above arguments, (3.32), and the boundedness of $\left\{\left\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)\right\|\right\}$ that

$$\sum_{j=0}^{k-1} A_j^k \leq M_2$$

for some constant $M_2 > 0$. Combining with (3.33) we come up to

$$\|p_j^{xk}\| \leq (1 + M_1 h_k)\|p_{j+1}^{xk}\| + A_j^k + \|\chi_j^k\| \tag{3.34}$$

for all $j = 0, \ldots, k-1$.

Using the induction method we can show that

$$\begin{aligned}\|p_j^{xk}\| &\le (1+M_1h_k)^{k-j}\|p_k^{xk}\| + \sum_{i=j}^{k-1} A_i^k(1+M_1h_k)^{i-j}\\ &\le e^{M_1} + e^{M_1}\sum_{i=0}^{k-1} A_i^k \le e^{M_1}(1+M_2)\end{aligned}$$

for $j = 2, \ldots, k-1$. The boundedness of p_0^{xk} and p_1^{xk} follows from (3.34) and the boundedness of $\{p_j^{xk}\}_{2\le j\le k}$. Thus we justify the boundedness of $\{(p_0^{xk}, \ldots, p_k^{xk})\}_{k\in I\!N}$.

To deal with the functions $q^{xk}(\cdot)$, we derive from their construction and the equations in (2.81) that

$$\begin{aligned}\sum_{j=0}^{k-1}\|q^{xk}(t_{j+1}) - q^{xk}(t_j)\| \le \left\|\chi_0^k\right\| + \left\|\chi_1^k\right\| + \lambda^k \sum_{j=0}^{k-1}\|h_k w_j^{xk}\|\\ + h_k\sum_{j=0}^{k-1}\left\|\nabla_x f\big(\bar{x}_j^k, \bar{a}_j^k\big)^*\big(\lambda^k(\theta^{xk}(t_j) - p_{j+1}^{xk})\big)\right\| \\ + \sum_{j=0}^{k-1}\|\nabla_x f(\bar{x}_j^k, \bar{a}_j^k)^*(\lambda^k h_k v_j^{xk})\| + \int_0^T\left\|\sum_{i=1}^m \gamma_i^k(t)x_i^*\right\| dt.\end{aligned} \tag{3.35}$$

It comes from (3.32) that the first term on the right-hand side of (3.35) is bounded by $\lambda_k\widetilde{L}$. We also have

$$h_k\sum_{j=0}^{k-1}\|\nabla_x f\big(\bar{x}_j^k, \bar{a}_j^k\big)^*\big(\lambda^k(\theta^{xk}(t_j) - p_{j+1}^{xk})\big)\| \le T\max_{0\le j\le k-1}\left\{\left\|\nabla_x f\big(\bar{x}_j^k, \bar{a}_j^k\big)^*\big(\lambda^k(\theta^{xk}(t_j) - p_{j+1}^{xk})\big)\right\|\right\},$$

which ensures the boundedness of the second term on the right-hand side of (3.35) by the boundedness of $\{p_j^{xk}\}_{k\in I\!N}$. Similarly we get the boundedness of the third term on the right-hand side of (3.35), while this property of the fourth term therein follows from (3.30). This shows by estimate (3.35) and the construction of $q^{xk}(t)$ on $[0, T]$ that the functions $q^{xk}(\cdot)$ are of uniformly bounded variation on this interval. In the same way we verify that $q^{uk}(\cdot)$ and $q^{ak}(\cdot)$ are of uniformly bounded variation on $[0, T]$ and arrive therefore at this conclusion for the whole triple $q^k(\cdot)$. It clearly implies that

$$2\|q^k(t)\| - \|q^k(0)\| - \|q^k(T)\| \le \|q^k(t) - q^k(0)\| + \|q^k(T) - q^k(t)\| \le \mathrm{var}\,(q^k; [0,T])$$

for all $t \in [0,T]$, which justifies the uniform boundedness of $q^k(\cdot)$ on $[0,T]$ is since both sequences $\{q^k(0)\}$ and $\{q^k(T)\}$ are bounded by (3.30). Then the classical Helly selection theorem allows us to find a function of bounded variation $q(\cdot)$ such that $q^k(t) \to q(t)$ as $k \to \infty$ pointwise on $[0,T]$. Employing further (3.30) and the measure construction in (3.29) tell us that the measure sequences $\{\gamma^k\}$ and $\{\xi^k\}$ are bounded in $C^*([0,T];\mathbb{R}^n)$ and $C^*([0,T];\mathbb{R})$ respectively. It follows from the weak* sequential compactness of the unit balls in these spaces that there are measures $\gamma \in C^*([0,T];\mathbb{R}^n)$ and $\xi \in C^*([0,T];\mathbb{R})$ such that the pair (γ^k, ξ^k) weak* converges to (γ, ξ) along some subsequence.

Combining the uniform boundedness of $q^k(\cdot)$, $w^k(\cdot)$, and $v^k(\cdot)$ on $[0,T]$ with (3.24), (3.26)–(3.28), and (3.30) allows us to deduce that the sequence $\{p^k(\cdot)\}$ is bounded in $W^{1,2}([0,T];\mathbb{R}^{3n})$ and hence weakly compact in this space. By the Mazur weak closure theorem we find $p(\cdot) \in W^{1,2}([0,T];\mathbb{R}^{3n})$ such that a sequence of convex combinations of $\dot{p}^k(t)$ converges to $\dot{p}(t)$ for a.e. $t \in [0,T]$. Passing now to the limit in (3.26)–(3.28) as $k \to \infty$ and using (3.17), we arrive at the representation of $\dot{p}(\cdot)$ in (3.5).

Next we proceed with deriving adjoint relationships involving the limiting function $q(\cdot)$ of bounded variation on $[0,T]$. Note to this end that if $\eta_i(t) > 0$ for some $t \in [0,T]$ and $i \in \{1,\ldots,m\}$, then $\eta_i^k(t) > 0$ whenever k is sufficiently large due to the a.e. convergence $\eta_i^k(\cdot) \to \eta_i(\cdot)$ on $[0,T]$. This implies by (2.89) that $\langle x_i^*, -q^{xk}(\nu(t)+h_k)+\lambda^k(\theta^{xk}(t)+v^{xk}(t))\rangle = 0$ for such k and t, and so we arrive at $\langle x_i^*, -q^x(t) + \lambda v^x(t)\rangle = 0$ while $k \to \infty$, which thus justifies the second implication in (3.8).

Remembering the construction of $q^k(\cdot)$ in Step 3 allows us to rewrite (2.82) and (2.83) as, respectively,

$$q^{uk}\big(\nu(t)+h_k\big)=\lambda^k\big(v^{uk}(t)+\theta^{uk}(t)\big) \text{ and } q^{ak}\big(\nu(t)+h_k\big)=\lambda^k\big(v^{ak}(t)+\theta^{ak}(t)\big) \qquad (3.36)$$

for $t\in(t_j^k,t_{j+1}^k)$ and $j=0,\ldots,k-1$. Passing to the limit in (3.36) with taking into account (3.3) and the assumptions on ℓ_2,ℓ_3 in (3.1), we arrive at both equations in (3.6). Observe further that

$$\left\|\int_{[t,T]}\sum_{i=1}^m\gamma_i^k(s)x_i^*ds-\int_{[t,T]}d\gamma(s)\right\|=\left\|\int_{[t,T]}d\gamma^k(s)-\int_{[t,T]}d\gamma(s)\right\|\to 0 \text{ as } k\to\infty$$

for all $t\in[0,T]$ except a countable subset of $[0,T]$ by the weak* convergence of the measures γ^k to γ in the space $C^*([0,T];\mathbb{R}^n)$; cf. [52, p. 325] for similar arguments. This ensures by (3.29) that

$$\int_{[t,T]}\sum_{i=1}^m\gamma_i^k(s)x_i^*ds\to\int_{[t,T]}d\gamma(s) \text{ as } k\to\infty. \qquad (3.37)$$

To obtain (3.7) by passing to the limit in (3.25), consider next the estimate

$$\begin{aligned}&\left\|\int_{[t,T]}\xi^k(s)\bar u^k\big(\nu^k(s)\big)ds-\int_{[t,T]}\bar u(s)d\xi(s)\right\|\\ \le&\left\|\int_{[t,T]}\xi^k(s)\bar u^k\big(\nu^k(s)\big)ds-\int_{[t,T]}\xi^k(s)\bar u(s)ds\right\|+\left\|\int_{[t,T]}\xi^k(s)\bar u(s)ds-\int_{[t,T]}\bar u(s)d\xi(s)\right\|\\ =&\left\|\int_{[t,T]}\xi^k(s)\big[\bar u^k\big(\nu^k(s)\big)-\bar u(s)\big]ds\right\|+\left\|\int_{[t,T]}\xi^k(s)\bar u(s)ds-\int_{[t,T]}\bar u(s)d\xi(s)\right\|\end{aligned} \qquad (3.38)$$

and observe that the first summand in the rightmost part of (3.38) disappears as $k\to\infty$ due to the uniform convergence $\bar u^k(\cdot)\to\bar u(\cdot)$ on $[0,T]$ and the uniform boundedness of $\int_0^T|\xi^k(t)|dt$ by (3.30). The second summand therein also converges to zero for all $t\in[0,T]$

except some countable subset by the weak* convergence $\xi^k \to \xi$ in $C^*([0,T];\mathbb{R})$. Hence we get

$$\int_{[t,T]} \xi^k(s)\bar{u}^k(\tau^k(s))ds \to \int_{[t,T]} \bar{u}(s)d\xi(s) \;\text{ as }\; k \to \infty$$

and thus arrive at (3.7) by passing to the limit in (3.25). Finally at this step, observe that the implications (3.8) follow directly by passing to the limit in their discrete counterparts (2.87), (2.88), and (2.89).

Step 5: *Transversality conditions.* Let us first verify the right endpoint one (3.9). For all $k \in I\!N$ we have by the second condition in (2.84) and the normal cone representation from (2.7) that

$$p_k^{uk} + 2\xi_k^k \bar{u}_k^k = \sum_{i=1}^m \eta_{ki}^k x_i^* = \sum_{i \in I(\bar{x}_k^k - \bar{u}_k^k)} \eta_{ki}^k x_i^* \in N(\bar{x}_k^k - \bar{u}_k^k; C), \tag{3.39}$$

where $\eta_{ki}^k = 0$ for $i \in \{1,\ldots,m\} \backslash I(\bar{x}_k^k - \bar{u}_k^k)$. Denote $\vartheta_k := \sum_{i \in I(\bar{x}_k^k - \bar{u}_k^k)} \eta_{ki}^k x_i^*$ and observe that a subsequence of $\{\vartheta_k\}$ converges to some ϑ due to the boundedness of $\{\xi_k^k\}$ by (3.30) and the convergence of $\{p_k^{uk}\}$ and $\{\bar{u}_k^k\}$ established above. Furthermore, it follows from the robustness of the normal cone in (3.39), the convergence $\bar{x}_k^k - \bar{u}_k^k \to \bar{x}(T) - \bar{u}(T)$, and the inclusion $I(\bar{x}_k^k - \bar{u}_k^k) \subset I(\bar{x}(T) - \bar{u}(T))$ for large $k \in I\!N$ that $\vartheta \in N(\bar{x}(T) - \bar{u}(T); C)$. Similarly the inclusion in (2.84) tells us that

$$-p_k^{xk} - \vartheta_k \in \lambda^k \partial\varphi(\bar{x}_k^k) \;\text{ for all }\; k \in I\!N. \tag{3.40}$$

Passing now to the limit as $k \to \infty$ in (3.39), (3.40), inclusion (2.55) for ξ_k^k, and the second condition in (2.84) with taking into account the robustness of the subdifferential in (3.40), we arrive at the relationships

$$-p^x(T)-\vartheta\in\lambda\partial\varphi\big(\bar{x}(T)\big),\quad p^u(T)-\vartheta\in-2\bar{u}(T)N\big(\|\bar{u}(T)\|;[r-\tau,r+\tau]\big),\quad p^a(T)=0$$

with $\vartheta=\sum_{i\in I(\bar{x}(T)-\bar{u}(T))}\eta_i(T)x_i^*\in N(\bar{x}(T)-\bar{u}(T);C)$. This clearly verifies the transversality conditions at the right endpoint in (3.9) supplemented by (3.11).

To justify the left endpoint transversality (3.10), we deduce from (2.82) and (2.83) for $j=0$ as well as the conditions on γ_0^k in Theorem 2.8 and the coderivative definition (2.48) that

$$\begin{aligned}&p_0^{uk}+h_k\lambda^k w_0^{uk}+2\xi_0^k\bar{u}_0^k-\lambda^k(v_0^{uk}+\theta_0^{uk})\\&\qquad\qquad\in D^*G\Big(\bar{x}_0^k-\bar{u}_0^k,-\frac{\bar{x}_1^k-\bar{x}_0^k}{h_k}-f(\bar{x}_0^k,\bar{a}_0^k)\Big)\big(-p_1^{xk}+\lambda^k(\theta_0^{xk}+v_0^{xk})\big),\\&p_0^{ak}=\lambda^k(v_0^{ak}+\theta_0^{ak}-h_k\lambda^k w_0^{ak}-h_k\nabla_a f(\bar{x}_0^k,\bar{a}_0^k)^*\big(\lambda^k(\theta_0^{xk}+v_0^{xk})\ \text{ whenever }\ k\in I\!N.\end{aligned}$$

Now we can pass to the limit as $k\to\infty$ in these relationships by taking into account (2.55) for $j=0$, (2.39), the construction of $q^k(t_j^k)=p_j^k$, the convergence statements for w_0^k, v_0^k, θ_0^k established above as well as robustness of the normal cone and coderivative. This readily gives us (3.10).

Step 6: *Measure nonatomicity conditions.* To verify condition (a) therein without any restriction on $\tau\in[0,\overline{\tau}]$, pick $t\in[0,T)$ with $\langle x_i^*,\bar{x}(t)-\bar{u}(t)\rangle<0$ for $i=1,\ldots,m$ and by continuity of $(\bar{x}(\cdot),\bar{u}(\cdot))$ find a neighborhood V_t of t such that $\langle x_i^*,\bar{x}(s)-\bar{u}(s)\rangle<0$ whenever $s\in V_t$ and $i=1,\ldots,m$. This shows by the established convergence of the discrete optimal solutions that $\langle x_i^*,\bar{x}^k(t_j^k)-\bar{u}^k(t_j^k)\rangle<0$ if $t_j^k\in V_t$ for $i=1,\ldots,m$ for all $k\in I\!N$ sufficiently large. Then it follows from (2.87) that $\gamma^k(t)=0$ on any Borel subset V of V_t, and therefore $\|\gamma^k\|(V)=\int_V d\|\gamma^k\|=\int_V\|\gamma^k(t)\|dt=0$ by the construction of the measure γ^k in (3.29).

Passing now to limit as $k \to \infty$ and taking into account the measure convergence obtained in Step 3, we arrive at $\|\gamma\|(V) = 0$ and thus justify the first measure nonatomicity condition. The proof of the nonatomicity condition (b) for the measure ξ is similar provided the choice of $\tau \in (0, \overline{\tau})$.

Step 7: *General nontriviality condition.* Let us justify the nontriviality condition (3.13) for any $\tau \in [0, \overline{\tau}]$ under the assumptions made therein. Suppose on the contrary to (3.13) that $\lambda = 0$, $q^x(0) = 0, q^u(0) = 0$, and $p(T) = 0$, which yields $\lambda^k \to 0, q^{xk}(0) \to 0, q^{uk}(0) \to 0$, and $p^k(T) \to 0$ as $k \to \infty$.

Furthermore, we have by the construction of $\xi^k(\cdot)$ on $[0, T]$ in Step 3 the equalities

$$\int_0^T |\xi^k(t)|dt = \sum_{j=0}^{k-1} h_k \frac{|\xi_j^k|}{h_k} = \sum_{j=0}^{k-1} |\xi_j^k|.$$

Taking into account that $I_0\left(-p_{j+1}^{xk} + \lambda^k(h_k^{-1}\theta_{xj}^k + v_j^{xk})\right) \cup I_>\left(-p_{j+1}^{xk} + \lambda^k(h_k^{-1}\theta_{xj}^k + v_j^{xk})\right) \subset I(\bar{x}_j^k - \bar{u}_j^k)$ and that $\langle x_i^*, \bar{x}_j^k - \bar{u}_j^k\rangle = 0$ for all $i \in I(\bar{x}_j^k - \bar{u}_j^k)$ and $j = 0, \ldots, k-1$, it follows from (2.82) that

$$2\xi_j^k\langle \bar{u}_j^k, \bar{x}_j^k - \bar{u}_j^k\rangle = \langle p_{j+1}^{uk} - p_j^{uk} - h_k\lambda^k w_j^{uk}, \bar{x}_j^k - \bar{u}_j^k\rangle, \quad j = 0, \ldots, k-1. \tag{3.41}$$

Using now the first condition in (3.12) imposed on the initial data of (P^τ) and that $\|\bar{u}_j^k\| = r$ for all $j = j_\tau(k), \ldots, k$ while $r - \tau - \varepsilon_k \le \|\bar{u}_j^k\| \le r + \tau + \varepsilon_k$ for $j = 0, \ldots, j_\tau(k) - 1$ and small $\varepsilon_k < \tau$, we get

$$|\langle \bar{u}_j^k, \bar{x}_j^k\rangle| \le \|\bar{u}_j^k\| \cdot \|\bar{x}_j^k\| \le (r + 2\tau)l < (r - 2\tau)^2 < r^2 = \|\bar{u}_j^k\|^2,$$

which immediately implies the validity of the estimate

$$|\langle \bar{u}_j^k, \bar{x}_j^k - \bar{u}_j^k\rangle| = \left|\langle \bar{u}_j^k, \bar{x}_j^k\rangle - \|\bar{u}_j^k\|^2\right| > 0 \tag{3.42}$$

whenever k is sufficiently large. On the other hand, (3.42) follows directly from the alternative assumption in (3.12) imposed on the optimal solution to (P^τ). Employing (3.42) and the equalities in (3.38) gives us

$$2|\xi_j^k| \leq \left(\|p_{j+1}^{uk} - p_j^{uk}\| + h_k\lambda^k\|w_j^{uk}\|\right)\frac{\|\bar{x}_j^k - \bar{u}_j^k\|}{|\langle\bar{u}_j^k, \bar{x}_j^k - \bar{u}_j^k\rangle|}, \quad j = 0, \ldots, k-1. \tag{3.43}$$

The obvious boundedness of $\left\{\frac{\|\bar{x}_j^k - \bar{u}_j^k\|}{|\langle\bar{u}_j^k, \bar{x}_j^k - \bar{u}_j^k\rangle|}\right\}$ allows us to assume without loss of generality that $\frac{\|\bar{x}_j^k - \bar{u}_j^k\|}{|\langle\bar{u}_j^k, \bar{x}_j^k - \bar{u}_j^k\rangle|} \leq 1$ for $j = 0, \ldots, k-1$, and then we get from (3.43) that

$$2\sum_{j=0}^{k-1}|\xi_j^k| \leq \sum_{j=0}^{k-1}\|p_{j+1}^{uk} - p_j^{uk}\| + h_k\lambda^k\sum_{j=0}^{k}\|w_j^{uk}\| \tag{3.44}$$

The second sum in (3.44) disappears as $k \to \infty$ due to the assumptions on ℓ_1; see (3.32) in Step 4. To proceed with the first sum in (3.44), we have the estimates

$$\begin{aligned}
\sum_{j=0}^{k-1}\|p_{j+1}^{uk} - p_j^{uk}\| \leq& \sum_{j=1}^{k-1}\|p_{j+1}^{uk} - p_j^{uk}\| + \|p_1^{uk}\| + \|p_0^{uk}\| \\
\leq& \lambda^k\sum_{j=1}^{k-1}\left\|\frac{\theta_j^{uk} - \theta_{j-1}^{uk}}{h_k}\right\| + \lambda^k\frac{\|\theta_0^{uk}\|}{h_k} + \lambda^k\sum_{j=1}^{k-1}\|v_j^{uk} - v_{j-1}^{uk}\| + \lambda^k\|v_0^{uk}\| + \|p_0^{uk}\| \\
\leq& 2\lambda^k\sum_{j=1}^{k-1}\left\|\frac{\bar{u}_{j+1}^k - 2\bar{u}_j^k + \bar{u}_{j-1}^k}{h_k}\right\| + 2\lambda^k\sum_{j=1}^{k-1}\left\|\frac{\bar{u}(t_{j+1}^k) - 2\bar{u}(t_j^k) + \bar{u}(t_{j-1}^k)}{h_k}\right\| \\
&+\lambda^k\frac{\|\theta_0^{uk}\|}{h_k} + \lambda^k\sum_{j=1}^{k-1}\|v_j^{uk} - v_{j-1}^{uk}\| + \lambda^k\|v_0^{uk}\| + \|p_0^{uk}\| \\
\leq& 4\widetilde{\mu}\lambda^k + \lambda^k\frac{\|\theta_0^{uk}\|}{h_k} + \lambda^k\sum_{j=1}^{k-1}\|v_j^{uk} - v_{j-1}^{uk}\| + \lambda^k\|v_0^{uk}\| + \|p_0^{uk}\|,
\end{aligned} \tag{3.45}$$

where $\widetilde{\mu}$ is defined in Theorem 2.8. The running cost structure (3.1) and differentiability of ℓ_2 in $\dot{u}$ yield

$$v_j^{uk} = \nabla_{\dot{u}}\ell_2\left(t_j, \frac{\bar{u}_{j+1}^k - \bar{u}_j^k}{h_k}, \frac{\bar{a}_{j+1}^k - \bar{a}_j^k}{h_k}\right) \text{ for } j = 0, \ldots, k-1.$$

Then the third estimate in (3.2) ensures that

$$\sum_{j=1}^{k-1}\|v_j^{uk}-v_{j-1}^{uk}\|\le\sum_{j=1}^{k-1}L\left((t_{j+1}-t_j)+\left\|\frac{\bar{u}_{j+1}^k-2\bar{u}_j^k+\bar{u}_{j-1}^k}{h_k}\right\|\right)\le L(T+\widetilde{\mu}).$$

Deduce further from the definition of θ^{uk} in (2.61) the representation

$$\frac{\theta_0^{uk}}{h_k}=\frac{2(\bar{u}_1^k-\bar{u}_0^k)}{h_k}-\frac{2\big(\bar{u}(h_k)-\bar{u}(0)\big)}{h_k}$$

and observe that $\lambda^k\dfrac{\|\theta_0^{uk}\|}{h_k}\to 0$ as $k\to\infty$ due to second estimate in (2.27) and the assumption imposed on $\bar{u}(\cdot)$ in Theorem 2.8 via [10, Theorem 3.1]. We have furthermore that

$$\|v_0^{uk}\|=\left\|\nabla_{\dot{u}}\ell_2\left(0,\frac{\bar{u}_1^k-\bar{u}_0^k}{h_k},\frac{\bar{a}_1^k-\bar{a}_0^k}{h_k}\right)\right\|\le L\left\|\frac{\bar{u}_1^k-\bar{u}_0^k}{h_k}\right\|\le L\widetilde{\mu}$$

due to (2.27) and the second estimate in (3.2). This shows therefore that

$$\sum_{j=0}^{k-1}\|p_{j+1}^{uk}-p_j^{uk}\|\to 0\ \text{ and }\ \int_0^T|\xi^k(t)|dt\to 0\ \text{ as }\ k\to\infty \tag{3.46}$$

by (3.45) and (3.44), respectively. Appealing again to (2.82) gives us

$$\begin{aligned}\int_0^T\left\|\sum_{i=1}^m\gamma_i^k(t)x_i^*\right\|dt=\sum_{j=0}^{k-1}\left\|h_k\sum_{i=1}^m\gamma_{ji}^kx_i^*\right\|\le\sum_{j=0}^{k-1}\|p_{j+1}^{uk}-p_j^{uk}\|+\lambda^kh_k\sum_{j=0}^{k-1}\|w_j^{uk}\|\\+2\sum_{j=0}^{k-1}|\xi_j^k|\to 0\end{aligned} \tag{3.47}$$

as $k\to\infty$ by the relationships in (3.32) and (3.46).

We next show that $q^{xk}(h_k)=p_1^{xk}\to 0$ as $k\to\infty$. Indeed, it follows from (2.81) that

$$\begin{aligned}\big\|q^{xk}(h_k)\big\|\le\big\|p_0^{xk}\big\|+\lambda^kh_k\big\|w_0^{xk}\big\|+\big\|\chi_0^k\big\|+h_k\big\|\nabla_xf(\bar{x}_0^k,\bar{a}_0^k)^*(\lambda^k(\theta^{xk}(0)-p_1^{xk}))\big\|\\+\int_0^T\left\|\sum_{i=1}^m\gamma_i^k(t)x_i^*\right\|dt,\end{aligned}$$

which verifies that $\lim_{k\to\infty}q^{xk}(h_k)=0$ due to (3.47) and the fact that the other quantities in the right hand side of the above estimate converge to 0 as $k\to\infty$.

To get a contradiction with our assumption on the violation of (3.13), it remains by (3.30) to verify that $\xi_k^k \to 0$ as $k \to \infty$. To see this, observe that the convergence $p^k(T) \to 0,\ \lambda^k \to 0$ implies by the first condition in (2.53) that $p_k^{xk} \to 0$, $p_k^{uk} \to 0$, and $\sum_{i=1}^m \eta_{ki}^k x_i^* \to 0$ as $k \to \infty$. Then it follows from the second condition in (2.53) that $\xi_k^k \bar{u}_k^k \to 0$, which yields $\xi_k^k \to 0$ since $\bar{u}_k^k \neq 0$ for large $k \in N$ due to $\bar{u}_k^k \to \bar{u}(T)$ and the assumptions on either $\bar{u}(T) \neq 0$ or $\tau < r$ that exclude vanishing $\bar{u}_k^k$ by the constraints in (2.9). Thus we arrive at a contradiction with (3.30) and so justify the nontriviality condition (3.13).

Step 8: *Enhanced nontriviality conditions.* Our final step is to justify the stronger/enhanced nontriviality conditions in (3.14), (3.15), and (3.16) under the interiority assumptions imposed therein provided that $0 < \tau < r$.

Consider first the left endpoint case (3.14) and suppose by contradiction that $(\lambda, q^u(0), p(T)) = 0$ under the assumption $\langle x_i^*, x_0 - \bar{u}(0)\rangle < 0$ for $i = 1, \ldots, m$. It then follows from (3.5) that

$$\begin{cases} \dot{p}^x(t) = \nabla_x f(\bar{x}(t), \bar{a}(t))(-q^x(t)), \\ \dot{p}^a(t) = \nabla_a f(\bar{x}(t), \bar{a}(t))(-q^x(t)), \end{cases} \tag{3.48}$$

for a.e. $t \in [0, T]$. Moreover, using (3.6) and (3.7) gives us $p^a(t) = q^a(t) = 0$ and thus $\dot{p}^a(t) = 0$ for a.e. $t \in [0, T]$. Hence, $\nabla_a f(\bar{x}(t), \bar{a}(t))(-q^x(t)) = 0$ by (3.48), which justifies that $q^x(t) = 0$ for a.e. $t \in [0, T]$ due to the surjectivity of the Jacobian $\nabla_a f(x, a)$. It then turns out from (3.48) that $\dot{p}^x(t) = 0$, and thus $p^x(t) = p^x(T) = 0$ for all $t \in [0, T]$. Using (3.6) again yields $q^x(t) = \int_{[t,T]} d\gamma(s) = 0$ for almost all $t \in [0, T]$ except at most a countable subset $\mathcal{A}$. Consider the following possibilities:

- If $0 \notin \mathcal{A}$ then $q^x(0) = 0$.

- If $0 \in \mathcal{A}$ then using the measure nonatomicity condition gives us some number $\delta > 0$

such that $\int_{[0,\delta)} d\gamma(s) = 0$, and thus $q^x(0) = \int_{[\delta,T]} d\gamma(s)$. We are left two following cases:

- If $\delta \notin \mathcal{A}$ then clearly $q^x(0) = 0$.

- If $\delta \in \mathcal{A}$ then there exists some $\widetilde{\delta} \in (0,\delta)$ such that $\widetilde{\delta} \notin \mathcal{A}$ due to the countability of $\mathcal{A}$. On the other hand, $[0,\widetilde{\delta})$ is a Borel subset of $V_0 = [0,\delta)$ so

$$q^x(0) = \int_{[0,\widetilde{\delta})} d\gamma(s) + \int_{[\widetilde{\delta},T]} d\gamma(s) = 0 + \int_{[\widetilde{\delta},T]} d\gamma(s) = 0.$$

In conclusion, we can always argue that $q^x(0) = 0$ and hence we lead to the contradiction to (3.13), which justifies the validity of (3.14).

We next consider the left endpoint case (3.15) and suppose by contradiction that $(\lambda, p(T)) = 0$ under the validity of the interiority condition in (3.15). It follows from the latter that $1 - \tau - \varepsilon_k < \|\bar{u}^k(0)\| < 1 + \tau + \varepsilon_k$ for $i = 1, \ldots, m$ and k sufficiently large. Then we deduce from (2.87) and (2.55) that $\gamma_{0i}^k = 0$ and $\xi_0^k = 0$ for $i = 1, \ldots, m$. Combining with (2.82) and the construction of $q^{uk}(\cdot)$ in Step 3 yields

$$q^{uk}(0) = p_0^{uk} = p_1^{uk} - h_k \lambda^k w_0^{uk}, \quad k \in I\!N.$$

Since $p_1^{uk} \to 0$ as $k \to \infty$ by the above proof, we conclude that $q^u(0) = \lim_{k\to\infty} q^{uk}(0) = 0$. Using the same argument as the previous case we can argue that $q^x(0) = 0$ and thus lead to the contradiction to the nontriviality condition (3.13), which verifies (3.15).

The justification of (3.16) comes from the interior assumptions imposed in (3.16), (3.8), and the transversality condition (3.9), and therefore we complete the proof of the theorem.

□

Let us now specify the general necessary optimality conditions of Theorem 3.2 to the important novel case of our consideration in this paper, where we have controls only in perturbations while u-controls in (P^τ) are *fixed*. Such a setting is used in Section 5 for applications to the controlled crowd motion model. In this case each problem (P) reduces to the following form $(\widetilde{P})$:

$$\text{minimize } \widetilde{J}[x,a] := \varphi(x(T)) + \int_0^T \ell\big(t, x(t), a(t), \dot{x}(t), \dot{a}(t)\big)\,dt$$

subject to the sweeping differential inclusion

$$-\dot{x}(t) \in N\big(x(t) - \bar{u}(t); C\big) + f\big(x(t), a(t)\big) \text{ a.e. } t \in [0,T], \quad x(0) := x_0 \in C \tag{3.49}$$

with the convex polyhedron C in (1.7) and the implicit state constraints

$$\big\langle x_i^*, x(t) - \bar{u}(t)\big\rangle \le 0 \text{ for all } t \in [0,T] \text{ and } i = 1, \ldots, m,$$

which follow from (3.49). As above, we study problem $(\widetilde{P})$ in the class of $(x(\cdot), a(\cdot)) \in W^{1,2}([0,T]; \mathbb{R}^{n+d})$. Observe that we do not need to consider in this case the τ-parametric version of $(\widetilde{P})$.

The next result follows from the specification of Theorem 3.2 and its proof in the case of $(\widetilde{P})$ by taking into account the structures of the sweeping set $C(t)$ and the running cost ℓ therein.

Corollary 3.3 (necessary conditions for sweeping optimal solutions with controlled perturbations). *Let $(\bar{x}(\cdot), \bar{a}(\cdot))$ be a given r.i.l.m. for $(\widetilde{P})$ satisfying* (H1), (H2), *all the appropriate assumptions of Theorem* 2.8, *and the assumptions on the running cost ℓ from Theorem* 3.2 *with $\ell_2 = 0$. Then there exist a number $\lambda \ge 0$, subgradient functions $w(\cdot) = (w^x(\cdot), w^a(\cdot)) \in L^2([0,T]; \mathbb{R}^{n+d})$ and $v(\cdot) = (v^x(\cdot), v^a(\cdot)) \in L^2([0,T]; \mathbb{R}^{n+d})$ well defined at $t = 0$ and satisfying* (3.3), *an adjoint arc $p(\cdot) = (p^x(\cdot), p^a(\cdot)) \in W^{1,2}([0,T]; \mathbb{R}^{n+d})$,* *and a measure* $\gamma = (\gamma_1, \ldots, \gamma_n) \in C^*([0,T];\mathbb{R}^n)$ *on* $[0,T]$ *such that we have conditions* (3.4), (3.8) *with the functions* $\eta_i \in L^2([0,T];\mathbb{R}_+)$ *uniquely defined by representation* (3.4) *together with the following relationships for a.e.* $t \in [0,T]$*:*

$$\begin{cases} \dot{p}^x(t) = \lambda w^x(t) + \nabla_x f\big(\bar{x}(t), \bar{a}(t)\big)^*\big(\lambda v^x(t) - q^x(t)\big), \\ \dot{p}^a(t) = \lambda w^a(t) + \nabla_a f\big(\bar{x}(t), \bar{a}(t)\big)^*\big(\lambda v^x(t) - q^x(t)\big), \end{cases} \tag{3.50}$$

where the vector function $q = (q^x, q^a)\colon [0,T] \to \mathbb{R}^n \times \mathbb{R}^d$ *is of bounded variation on* $[0,T]$ *satisfying*

$$q^a(t) \in \lambda \partial \ell_3\big(t, \dot{\bar{a}}(t)\big) \;\; \text{for a.e. } t \in [0,T] \quad \text{and} \tag{3.51}$$

$$q(t) := p(t) - \int_{[t,T]} \big(d\gamma(s), 0\big) \;\; \text{on} \;\; [0,T] \tag{3.52}$$

except at most a countable subset for its left-continuous representative. We also have the measure nonatomicity condition (a) *of Theorem* 3.2 *and the transversality relationships*

$$\begin{cases} -p^x(T) \in \lambda \partial\varphi\big(\bar{x}(T)\big) + \displaystyle\sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) x_i^* \subset \partial\varphi\big(\bar{x}(T)\big) + N\big((\bar{x}(T) - \bar{u}(T); C\big), \\ p^a(T) = 0, \quad \text{and} \;\; q^a(0) = \lambda v^a(0). \end{cases} \tag{3.53}$$

Finally, the enhanced nontriviality condition

$$\lambda + \|p(T)\| \neq 0 \tag{3.54}$$

holds provided that either $\langle \bar{x}(t), \bar{u}(t)\rangle \neq \|\bar{u}(t)\|^2$ *on* $[0,T)$*, or* $\langle x_i^*, x_0 - \bar{u}(0)\rangle < 0$ *for all* $i = 1, \ldots, m$ *and the Jacobian* $\nabla_a f(x,a)$ *is surjective.*

Remark 3.4 (discussions on optimality conditions). The results of Theorem 3.2 and Corollary 3.3 provide comprehensive necessary optimality conditions for a broad class of intermediate (between weak and strong with including the latter) local minimizers of state-constrained sweeping control problems concerning highly unbounded and non-Lipschitzian differential inclusions. Now we briefly discuss some remarkable features of the obtained results with their relationships to known results in this direction.

(i) As has been well recognized in standard optimal control theory for differential equations and Lipschitzian differential inclusions with state constraints, necessary optimality conditions for such problems may exhibit the *degeneration phenomenon* when they hold for every feasible solution with some nontrivial collection of dual variables. In particular, this could happen if the initial vector at $t = 0$ belongs to the boundary of state constraints; see [4, 52] for more discussions and references.

It may also be the case for our problem (P^τ) under the general nontriviality condition (3.13) when, e.g., $\tau = 0$ and the vector $x_0 - \bar{u}(0)$ lays on the boundary of the polyhedral set C. However, the degeneration phenomenon is surely excluded in (P^τ) by the enhanced nontriviality in (3.14), (3.15), (3.16), and by the condition $\langle \bar{x}(t), \bar{u}(t) \rangle \neq \|\bar{u}(t)\|^2$ on $[0, T)$ in $(\widetilde{P})$ even in the case where either $x_0 - \bar{u}(0)$ or $\bar{x}(T) - \bar{u}(T)$ is a boundary point of the convex polyhedron C; see Examples 4.2, 4.3 and also Examples 5.1, 5.2 for the crowd motion model. As other examples demonstrate (see, in particular, Example 4.1), even in the case of potential degeneracy as in (3.13) for $\tau = 0$ under the violation of the aforementioned conditions that rule out the degeneration phenomenon, the obtained results can be useful to determine optimal solutions.

(ii) Let us draw the reader's attention to some specific features of the new *transversality conditions* obtained in Theorem 3.2 and Corollary 3.3. The transversality condition at the *left endpoint* in (3.10) and (3.53) may look surprising at the first glance since the initial vector x_0 of the feasible sweeping trajectories $x(\cdot)$ is fixed. However, it is not the case for control functions $u(\cdot)$ and $a(\cdot)$, which are incorporated into the differential inclusion (2.7) and the cost functional (1.5) with their initial points being reflected in (3.10). The usage of the left transversality condition (3.10) allows us to exclude in Example 4.1 the potential degeneration term $q^u(0)$ from the general nontriviality condition (3.13) and then to calculate an optimal solution to the sweeping control problem under consideration.

Observe that we get the two types of the transversality conditions for $p(T)$ at the *right endpoint* in (3.9) and (3.53): one expressed directly via $\eta_i(T)$ and other given via the normal cone $N(\bar{x}(T)-\bar{u}(T);C)$ due to (3.11). While the second type of transversality is more expected, the first type is essentially more precise. Indeed, the normal cone transversality may potentially lead us to degeneration when $\bar{x}(T)-\bar{u}(T)$ lays at the boundary of C. On the other hand, degeneration is completely excluded in this case if we have $\eta_i(T)=0$ as $i\in I(\bar{x}(T)-\bar{u}(T))$ for the endpoint vectors $\eta_i(T)$, which may occur independently of their *a priori* location at $N(\bar{x}(T)-\bar{u}(T);C)$ due to the fact that the vectors $\eta_i(T)$ are *uniquely determined* by representation (3.4) of the term $-\dot{\bar{x}}(T)-f(\bar{x}(t),\bar{a}(t))$ via the linearly independent generating vectors x_i^*. This is explicitly illustrated by Example 4.1.

(iii) It has been largely understood in optimal control of differential equations and Lipschitzian differential inclusions that necessary optimality conditions for problems with inequality state constraints are described via *nonnegative Borel measures*. In the case of (P^τ) we have both inequality and equality state constraints on $z(\cdot)$ given by (2.9) and (1.8) that are reflected in Theorem 3.2 by the measure $\xi\in C^*([0,T];\mathbb{R})$ and $\gamma\in C^*([0,T];\mathbb{R}^m)$, respectively. In problem $(\widetilde{P})$ we do not have state constraints for the u-components, and so only the measure γ appears in the optimality conditions of Corollary 3.3. But even in the latter case we *do not ensure the nonnegativity* of γ (see Examples 5.1 and 5.2 for the controlled crowd motion model), which once more reveals a significant difference between the sweeping control problems governed in fact by *evolution/differential variational inequalities* from the conventional state-constrained control problem considered in the literature. On the other hand, all the examples presented in Sections 4 and 5 show that our results agree with those known for conventional models while indicating that the corresponding measures become nonzero at the points where optimal trajectories *hit the boundaries* of state constraints and *stay such* on these boundaries; see Examples 5.1 and 5.2 to illustrate the latter phenomenon.

(iv) Finally, we compare the results derived in Theorem 3.2 (and their consequences in Corollary 3.3) with the most recent necessary optimality conditions obtained in [13, Theorem 6.1] for problem $(\bar{P}^\tau)$ as $0<\tau<T$ of minimizing the cost functional

$$\bar{J}[x,u,b]:=\varphi\big(x(T)\big)+\int_0^T \ell\big(t,x(t),u(t),b(t),\dot{x}(t),\dot{u}(t),\dot{b}(t)\big)dt$$

over absolutely continuous controls $u=(u_1,\ldots,u_m)\colon[0,T]\to\mathbb{R}^{mn}$, $b=(b_1,\ldots,b_m)\colon[0,T]\to\mathbb{R}^m$ and the corresponding absolutely continuous trajectories $x\colon[0,T]\to\mathbb{R}^n$ of the unperturbed sweeping inclusion

$$-\dot{x}(t)\in N\big(x(t);C(t)\big)\ \text{ for a.e. }\ t\in[0,T],\quad x(0):=x_0\in C(0),$$

with the sweeping set $C(t)$ and the constraints on u-controls given by

$$C(t):=\big\{x\in\mathbb{R}^n\big|\ \langle u_i(t),x\rangle\le b_i(t)\ \text{ for all }\ i=1,\ldots,m\big\},$$

$$\|u_i(t)\|=1\ \text{ for all }\ t\in[0,T]\ \text{ and }\ i=1,\ldots,m.$$

We can see that problem $(\bar{P}^\tau)$ is different from (P^τ) by the *absence of controlled perturbations* (which is of course the underlying feature of our problem (P^τ) and its applications to the crowd motion model), the choice of τ, and a bit different class of feasible solutions. On the other hand, while ignoring these differences, problem (P^τ) can be reduced to $(\bar{P}^\tau)$ with no u-controls (they are replaced by the generating vectors x_i^* of the polyhedron C) and with b-controls given in the form

$$b_i(t) := \langle x_i^*, u(t)\rangle \text{ for all } t \in [0,T] \text{ and } i = 1,\ldots,m$$

via the u-controls in (P^τ). However, problem $(\bar{P}^\tau)$ obtained in this way from (P^τ) in the absence of perturbations is not considered in [13], since we do have the *pointwise constraints* on $u_i(t)$ in (2.9), which are in fact a part of the state constraints on $z(t)$ in the setting of (2.7) under investigation, while there are *no any constraints* on $b_i(t)$ in [13]. Necessary optimality conditions for problems $(\bar{P}^\tau)$ of this type (where τ does not play any role since the u-controls are fixed) are specified in [13, Theorem 6.3]. It is not hard to check that the results obtained therein are included in those established in Theorem 3.2 for (P^τ) in the case where both problems are the same. However, even in this (not so broad) case we obtain additional information in Theorem 3.2 and Corollary 3.3 in comparison with [13]. Let us list the main new ingredients of our results for (P^τ) in the common setting with [13, Theorem 6.3] and also in a similar (while different) setting of [13, Theorem 6.1] for $(\bar{P}^\tau)$ with u-control components, which can be incorporated therein by using the *more precise* discrete approximation technique developed in this paper:

- The new transversality conditions at the left endpoint; see remark (ii) above.
- Both types of transversality at the right endpoint discussed in remark (ii) are different

and more convenient for applications in comparison with (6.10)–(6.12) in [13]. Observe that the latter ones are given implicitly as equations for p^x, p^u, p^b at the local optimal solution $\bar{z}(T)$.

• Our results are applied to the general case of the parameter τ and its interrelation with another parameter r in the u-control bounds in contrast to only the interior case of $\tau \in (0, T)$ with $r = 1$ in [13].

• Our general nontriviality condition (3.13) contains only the u-component $q^u(0)$ and x-component $q^x(0)$ in contrast to all the components of $q(0)$ in the corresponding condition $\lambda + \|p(T)\| + \|q(0)\| \neq 0$ of [13].

• Theorem 3.2 and Corollary 3.3 present more conditions that surely rule out the degeneracy phenomenon in comparison with the corresponding results of [13, Theorems 6.1, 6.3]; see the discussion in remark (i). Note that the appearance of degeneracy is also excluded by the new transversality conditions as discussed in remark (ii) and illustrated by the examples below.

• The presence of controlled perturbations in (P^τ) and $(\widetilde{P})$ allows us to reveal new behavior phenomena for the measure γ responsible for the state constraints (1.8) in comparison with the settings of [13], even in the absence of the measure ξ responsible for the u-constraints in (2.9); see remark (iii). In particular, Examples 5.1 and 5.2 illustrate behavior of the measure γ in keeping the optimal trajectory on the boundary of state constraints in the crowd motion model.

CHAPTER 4 NUMERICAL EXAMPLES

In this section we present three academic examples illustrating some characteristic features of the obtained necessary optimality conditions for problems (P^τ) and $(\widetilde{P})$ and their usefulness to determine optimal solutions and exclude nonoptimal ones in rather simple settings. More involved examples with our major applications to the crowd motion model in a corridor are given in Section 5.

Example 4.1 (optimal controls in both sweeping set and perturbations). Consider problem (P^τ) with any $0 \le \tau < 1/2$ and the following data:

$$\begin{cases} n = m = d = T = 1,\ x_0 := 0,\ x_1^* := 1,\ f(x,a) := a,\ \varphi(x) := \dfrac{(x-1)^2}{2}, \\ \ell(t,x,u,a,\dot{x},\dot{u},\dot{a}) := \dfrac{1}{2}a^2. \end{cases} \tag{4.1}$$

Figure 1: Direction of optimal control

In this case we have $C = \mathbb{R}_-$. The structure of the cost functional in (4.1) allows us to assume without loss of generality that a-controls are uniformly bounded, and thus (P^τ) admits an optimal solution $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot)) \in W^{1,2}([0,1];\mathbb{R}^3)$ by [10, Theorem 4.1]. It is also easy to see that all the assumptions of Theorem 3.2 are satisfied. Furthermore, it follows from the structure of (P^τ) with $r = 1/2$ in (2.9) that $\bar{u}(t) = 1/2$ on $[\tau, 1-\tau]$ and $\bar{u}(t) \in [1/2-\tau, 1/2+\tau]$ on $[0,\tau) \cup (1-\tau, 1]$; see Figure 1. Supposing further that $\bar{x}(t) \in \operatorname{int}(C + \bar{u}(t))$ for any $t \in [0,1)$ and that $-\dot{\bar{x}}(1) = f(\bar{x}(1), \bar{a}(1))$, we see that these assumptions are realized for the optimal solution found via the necessary optimality conditions of Theorem 3.2.

With taking into account that the second assumption in (3.12) holds in our case, we get from Theorem 3.2 the following relationships, where $\lambda \ge 0$ and $\eta(\cdot) \in L^2([0,1];\mathbb{R}_+)$ being well defined at $t=1$:

(1) $w(t)=\big(0,0,\bar{a}(t)\big),\ v(t)=0$ for a.e. $t\in[0,1]$;

(2) $\bar{x}(t)<\bar{u}(t)\Longrightarrow\eta(t)=0$ for a.e. $t\in[0,1)$;

(3) $\eta(t)>0\Longrightarrow q^x(t)=0$ for a.e. $t\in[0,1]$ including $t=1$;

(4) $-\dot{\bar{x}}(t)=\eta(t)+f\big(\bar{x}(t),\bar{a}(t)\big)=\eta(t)+\bar{a}(t)$ for a.e. $t\in[0,1]$ including $t=1$;

(5) $\big(\dot{p}^x(t),\dot{p}^u(t),\dot{p}^a(t)\big)=\big(0,0,\lambda\bar{a}(t)-q^x(t)\big)$ for a.e. $t\in[0,1]$;

(6) $q^u(t)=0,\ q^a(t)=0$ for a.e. $t\in[0,1]$;

(7) $\big(q^x(t),q^u(t),q^a(t)\big)=\big(p^x(t),p^u(t),p^a(t)\big)-\Big(\int_{[t,1]}d\gamma,\int_{[t,1]}2d\xi-d\gamma,0\Big)$ for a.e. $t\in[0,1]$;

(8) $-p^x(1)=\lambda\big(\bar{x}(1)-1\big)+\eta(1),\ p^a(1)=0$;

(9) $p^u(1)\in\eta(1)+2\bar{u}(1)N\big(\bar{u}(1);[1/2-\tau,1/2+\tau]\big)$;

(10) $\eta(1)\in N\big(\bar{x}(1)-\bar{u}(1);C\big)$;

(11) $q^u(0)\in-2\bar{u}(0)N\big(\bar{u}(0);[1/2-\tau,1/2+\tau]\big)+D^*G\big(x_0-\bar{u}(0),-\dot{\bar{x}}(0)-\bar{a}(0)\big)\big(-q^x(0)\big)$;

(12) $\lambda+|q^u(0)|+|p(1)|\neq 0$.

Since $\bar{x}(t)-\bar{u}(t)\in\operatorname{int}C$ for all $t\in[0,1)$, the coderivative of the mapping $G(\cdot)=N(\cdot;C)$ disappears in the left endpoint transversality condition (11). Furthermore, we have $\eta(1)=0$ by (3) due to the assumption $-\dot{\bar{x}}(1)=f(\bar{x}(1),\bar{a}(1))$ imposed on the optimal solution. Hence condition (10) holds automatically, and we arrive at the updated transversality relationships:

$$\begin{cases} -p^x(1) = \lambda\big(\bar{x}(1) - 1\big),\ p^u(1) \in 2\bar{u}(1)N\big(\bar{u}(1);[1/2-\tau, 1/2+\tau]\big), \\ \\ q^u(0) \in -2\bar{u}(0)N\big(\bar{u}(0);[1/2-\tau, 1/2+\tau]\big). \end{cases} \tag{4.2}$$

It follows from (5)–(7) that $p^x(\cdot)$ is a constant function on $[0,1]$ and that

$$\lambda\bar{a}(t) = q^x(t) = p^x(1) - \int_{[t,1]} d\gamma \ \text{ for a.e. } \ t \in [0,1]. \tag{4.3}$$

The next assertion that holds in any finite-dimensional space is a consequence of the measure nonatomicity condition (a) of Theorem 3.2, which is essential in this and other examples.

Claim: *Let $\langle x^*, \bar{x}(s) - \bar{u}(s)\rangle < 0$ for all $s \in [t_1, t_2]$ with $t_1, t_2 \in [0, T)$ and some vector $x^* \in \mathbb{R}^n$ under the validity of the measure nonatomicity condition (a) of Theorem 3.2 involving the vector x^* and the measure γ therein. Then $\gamma([t_1, t_2]) = 0$ and $\gamma(\{s\}) = 0$ whenever $s \in [t_1, t_2]$. Thus we also have $\gamma((t_1, t_2)) = \gamma([t_1, t_2)) = \gamma((t_1, t_2]) = 0$.*

To verify this claim, pick any $s \in [t_1, t_2]$ with $\langle x_1^*, \bar{x}(t) - \bar{u}(t)\rangle < 0$ and find by the imposed measure nonatomicity condition a neighborhood V_s of s in $[0, T]$ such that $\gamma(V) = 0$ for all Borel subsets V of V_s, and hence obviously $\gamma(\{s\}) = 0$. Since $[t_1, t_2] \subset \bigcup_{s\in[t_1,t_2]} V_s$ and $[t_1, t_2]$ is compact, there are finitely many points $s_1, \ldots, s_p \in [t_1, t_2]$ with $[t_1, t_2] \subset \bigcup_{i=1}^p V_{s_i}$. For each $i = 1, \ldots, p-1$ take $\widetilde{s}_i \in V_{s_i} \cap V_{s_{i+1}}$ such that $[s_i, \widetilde{s}_i] \subset V_{s_i}$ and $[\widetilde{s}_i, s_{i+1}] \subset V_{s_{i+1}}$, where $s_1 = t_1$ and $s_p = t_2$ without loss of generality. Then the claim readily follows from the equalities

$$\gamma([t_1, t_2]) = \gamma\left(\bigcup_{i=1}^{p-1} [s_i, \widetilde{s}_i) \cup [\widetilde{s}_i, s_{i+1})\right) = \sum_{i=1}^{p-1} \Big(\gamma([s_i, \widetilde{s}_i)) + \gamma([\widetilde{s}_i, s_{i+1}))\Big) = 0.$$

Going back to our example, observe that the validity of $\bar{x}(s) < \bar{u}(s)$ for all $s \in [t,1)$ with $t \in [0,1)$ yields $\gamma([t,1]) = \gamma(\{1\})$. Indeed, it follows from the above claim that for all large $k \in I\!N$ we get

$$\gamma([t,1]) = \gamma([t,1)) + \gamma(\{1\}) = \gamma\Big(\big[t, 1-k^{-1}\big] \cup \bigcup_{n\geq k} \big(1-n^{-1}, 1-(n+1)^{-1}\big] \Big) + \gamma(\{1\})$$
$$= \gamma\big(\big[t, 1-k^{-1}\big]\big) + \sum_{n\geq k} \gamma\Big(\big(1-n^{-1}, 1-(n+1)^{-1}\big]\Big) + \gamma(\{1\}) = \gamma(\{1\}).$$

This allows us to deduce from (4.3) that

$$\lambda \bar{a}(t) = p^x(1) - \gamma(\{1\}) \ \text{ for a.e. } \ t \in [0,1]. \tag{4.4}$$

To proceed further, consider first the case where $1/2 - \tau < \bar{u}(t) < 1/2 + \tau$ for $t = 0, 1$. In this case we have $q^u(0) = p^u(1) = 0$ by (4.2) and so $\lambda > 0$, since the opposite would contradict the nontriviality condition (12) by taking (8) with $\eta(1) = 0$ into account. It follows now from (4.4) that $\bar{a}(\cdot)$ must be a constant function, $\bar{a}(\cdot) \equiv \vartheta$ on $[0,1]$, due to its continuity. Then (2) and (4) ensure that

$$\bar{x}(t) = x_0 + \int_0^t \dot{\bar{x}}(s)ds = -\int_0^t \vartheta ds = -t\vartheta \ \text{ for all } \ t \in [0,1].$$

Consequently, the cost functional in our problem is calculated by

$$J[\bar{x}, \bar{u}, \bar{a}] = \frac{(-\vartheta - 1)^2}{2} + \frac{\vartheta^2}{2} = \vartheta^2 + \vartheta + \frac{1}{2}$$

and clearly achieves its absolute minimum at $\vartheta = -1/2$. Thus in this case we arrive by the necessary optimality conditions of Theorem 3.2 at the (local) optimal solution

$$\begin{cases} \bar{x}(t) = t/2, \ \ \bar{a}(t) = -1/2 \ \text{ on } \ [0,1], \ \ \bar{u}(t) = 1/2 \ \text{ on } \ [\tau, 1-\tau], \ \text{ and} \\ \bar{u}(t) \in (1/2 - \tau, 1/2 + \tau) \ \text{ on } \ [0,\tau) \cup (1-\tau, 1], \end{cases}$$

which satisfies all the preliminary assumptions imposed above.

In the case where $\bar{u}(t) \in \{1/2-\tau, 1/2+\tau\}$ as $t=0,1$ we get from (4.2) that

$$\begin{cases} p^u(1) \leq 0,\ q^u(0) \geq 0 & \text{if} \quad \bar{u}(0) = 1/2-\tau,\ \bar{u}(1) = 1/2-\tau, \\ p^u(1) \geq 0,\ q^u(0) \geq 0 & \text{if} \quad \bar{u}(0) = 1/2-\tau,\ \bar{u}(1) = 1/2+\tau, \\ p^u(1) \leq 0,\ q^u(0) \leq 0 & \text{if} \quad \bar{u}(0) = 1+\tau,\ \bar{u}(1) = 1-\tau, \\ p^u(1) \geq 0,\ q^u(0) \leq 0 & \text{if} \quad \bar{u}(0) = 1/2+\tau,\ \bar{u}(1) = 1/2+\tau, \end{cases}$$

which does not provide sufficient information to conclude that $p^u(1) = q^u(0) = 0$ and thus $\lambda > 0$. If the latter holds, we can proceed similarly to the interior case and find local minimizers as above. However, the case of $\lambda = 0$ remains open from the viewpoint of Theorem 3.2.

Observe finally that by fixing the u-control component as $\bar{u}(\cdot) = 1/2$ on $[0,1]$ we reduce problem (P^τ) of this example to the type of $(\widetilde{P})$. Then the necessary conditions of Corollary 3.3 and the arguments above allow us to calculate, by taking into account the existence result of [10, Theorem 4.1], the unique global optimal solution $(\bar{x}(t), \bar{a}(t)) = (t/2, -1/2)$ for all $t \in [0,1]$.

Example 4.2 (nonsmooth problems with controlled perturbations). Consider problem $(\widetilde{P})$ with $n, m, d, T, x_0, x_1^*, f(x,a)$ as in (4.1), fixed $\bar{u}(t) = r$ on $[0,1]$, and the cost functions $\varphi(x) := (x-1)^2$,

$$\ell(t,x,u,a,\dot{x},\dot{u},\dot{a}) := (a+2t)^2 + \alpha|\dot{a} + 4t - 1| \quad \text{for} \quad \alpha \geq 0. \tag{4.5}$$

Let us first examine the case of the parameters $r = 1$ and $\alpha = 0$. The structure of the cost functional in this case suggests a natural candidate for the optimal solution $(\bar{x}(t), \bar{a}(t)) = (t^2, -2t)$ on $[0,1]$. Observe that $\bar{x}(t) < \bar{u}(t)$ for all $t \in [0,1)$ and that $\bar{x}(t) \cdot \bar{u}(t) \neq |\bar{u}(t)|^2$ for all $t \in [0,1)$. Applying now the necessary optimality conditions of Corollary 3.3 gives us the following relationships with a number $\lambda \geq 0$ and a function $\eta(\cdot) \in L^2([0,1];\mathbb{R}_+)$ well defined at $t=1$:

(1) $w(t) = \big(0, 2(\bar{a}(t)+2t)\big),\ v(t) = (0,0)$ for a.e. $t \in [0,1]$;

(2) $\bar{x}(t) < \bar{u}(t) \Longrightarrow \eta(t) = 0$ for a.e. $t \in [0,1]$ including $t=1$;

(3) $\eta(t) > 0 \Longrightarrow q^x(t) = 0$ for a.e. $t \in [0,1]$;

(4) $-\dot{\bar{x}}(t) = \eta(t) + f\big(\bar{x}(t), \bar{a}(t)\big) = \eta(t) + \bar{a}(t)$ for a.e. $t \in [0,1]$;

(5) $\big(\dot{p}^x(t), \dot{p}^a(t)\big) = \big(0, 2\lambda(\bar{a}(t)+2t) - q^x(t)\big)$ for a.e. $t \in [0,1]$;

(6) $q^a(t) = 0$ for a.e. $t \in [0,1]$;

(7) $\big(q^x(t), q^a(t)\big) = \big(p^x(t), p^a(t)\big) - \left(\int_{[t,1]} d\gamma, 0\right)$ for a.e. $t \in [0,1]$;

(8) $-p^x(1) \in 2\lambda\big(\bar{x}(1)-1\big) + \eta(1),\ p^a(1) = 0$;

(9) $\eta(1) \in N\big(\bar{x}(1) - \bar{u}(1); C\big)$;

(10) $\lambda + |p^x(1)| \neq 0$.

Combining the relationships in (5)–(7) gives us the equation

$$2\lambda\big(\bar{a}(t)+2t\big) = q^x(t) = p^x(1) - \gamma([t,1]) \ \text{ for a.e. } \ t \in [0,1]. \tag{4.6}$$

Letting $\lambda > 0$ and taking into account that $x(t) - u(t) < 0$ for all $t \in [0,1)$, we get by the arguments in Example 4.1 that $\gamma([t,1]) = \gamma(\{1\})$. It implies that $\bar{a}(t) + 2t$ reduces to a constant ϑ a.e. on $[0,1]$, which ensures that $\bar{a}(t) = -2t + \vartheta$ for all $t \in [0,1]$ due to the continuity of $\bar{a}(\cdot)$. It follows from (4) that

$$\bar{x}(t) = \int_0^t \dot{\bar{x}}(s)ds = -\int_0^t \eta(s)ds - \int_0^t (-2s+\vartheta)ds = t^2 - t\vartheta, \quad t \in [0,1].$$

Thus $\bar{x}(1) = 1 - \vartheta$, which gives the value of $2\vartheta^2$ to the cost functional with the minimal value achieved at $\vartheta = 0$. This confirms via Corollary 3.3 the optimality of the solution $(\bar{x}(t), \bar{a}(t)) = (t^2, -2t)$ for the above choice of the parameters $(r, \alpha) = (1, 0)$ in the problem under consideration. Note that the other conditions in (1)–(10) besides those used above hold automatically for $(\bar{x}(\cdot), \bar{a}(\cdot))$ with $p^x(1) = \eta(1) = 0$.

Consider now this problem $(\widetilde{P})$ with the parameter values $r = 2$ and $\alpha > 0$; the latter generates nonsmoothness in (4.5). Let us check that the feasible solution $(\bar{x}(t), \bar{a}(t)) = (t^2, -2t)$ on $[0, 1]$ is not locally optimal anymore for $(\widetilde{P})$ by using the necessary optimality conditions of Corollary 3.3 listed above with the replacement of (6) by the subdifferential inclusion (3.51) in the nonsmooth case of (4.5). It follows from (7) that $q^a(t) = p^a(t)$ for a.e. $t \in [0, 1]$. Furthermore, (3.51) tells us in this case that $q^a(t) = -\alpha\lambda$ for a.e. $0 \le t < 3/4$ and $q^a(t) = \alpha\lambda$ for a.e. $3/4 < t \le 1$. Thus we have

$$p^a(t) = \begin{cases} -\alpha\lambda & \text{for all } \ 0 \le t < 3/4, \\ \alpha\lambda & \text{for all } \ 3/4 < t \le 1, \end{cases}$$

which yields by the continuity of $p^a(\cdot)$ on $[0, 1]$ that $\alpha\lambda = 0$ and hence $\lambda = 0$. Then it follows from (8) and (2) with $\bar{x}(1) = 1 < \bar{u}(1) = 2$ in that $p^x(1) = 0$. This contradicts the nontriviality condition (10) and hence justifies that the given pair $(\bar{x}(\cdot), \bar{a}(\cdot))$ fails to be an optimal solution to $(\widetilde{P})$ with $r = 2$ and $\alpha > 0$.

The the next example addresses a two-dimensional perturbed sweeping process and demonstrates the possibility to determine optimal solutions by using the necessary optimality conditions of Corollary 3.3.

Example 4.3 (two-dimensional sweeping process with controlled perturbations.)

Consider problem $(\widetilde{P})$ with the following initial data:

$$n = m = d = 2,\ T = 1,\ x_0 := (0,-1),\ x_1^* := (1,0),\ x_2^* := (0,1),\ f(x,a) := a,\ \varphi(x) := 0,$$

and $\ell(t,x,u,a,\dot{x},\dot{u},\dot{a}) := (\|\dot{x}\|^2 + \|a\|^2)/2$. Given $\bar{u}(\cdot) = (1,0)$ on $[0,1]$, apply the necessary optimality conditions of Corollary 3.3 to determine (local) optimal solutions $\bar{a}(\cdot) = (\bar{a}_1(\cdot), \bar{a}_2(\cdot))$ and $\bar{x}(\cdot) = (\bar{x}_1(\cdot), \bar{x}_2(\cdot))$ to this problem. We seek for solutions to $(\widetilde{P})$ such that

$$\langle x_i^*, \bar{x}(t) - \bar{u}(t)\rangle < 0 \ \text{ for all } \ t \in [0,1),\ i = 1,2, \quad \text{and} \quad \bar{x}(1) - \bar{u}(1) \in \operatorname{bd} C. \tag{4.7}$$

and show that (4.7) holds for $(\bar{x}(\cdot), \bar{a}(\cdot))$ found below by using the necessary optimality conditions of Corollary 3.3. In the case of $(\widetilde{P})$ under consideration these conditions look as follows, where $\lambda \geq 0$ and $\eta(\cdot) = \big(\eta_1(\cdot, \eta_2(\cdot)\big) \in L^2([0,1]; \mathbb{R}_+)$ well defined at $t = 1$:

(1) $w(t) = \big(0,0,\bar{a}(t)\big),\ v(t) = \big(\dot{\bar{x}}(t),0,0\big)$ for a.e. $t \in [0,1]$;

(2) $\langle x_i^*, \bar{x}(t) - \bar{u}(t)\rangle < 0 \Longrightarrow \eta_i(t) = 0$ for $i = 1,2$ and a.e. $t \in [0,1)$;

(3) $\eta_i(t) \Longrightarrow \langle x_i^*, \lambda\dot{\bar{x}}(t) - q^x(t)\rangle = 0$ for $i = 1,2$ and a.e. $t \in [0,1]$;

(4) $-\dot{\bar{x}}(t) = \big(-\dot{\bar{x}}_1(t), -\dot{\bar{x}}_2(t)\big) = (\eta_1(t), \eta_2(t)) + \big(\bar{a}_1(t), \bar{a}_2(t)\big)$ for a.e. $t \in [0,1]$;

(5) $\big(\dot{p}^x(t), \dot{p}^a(t)\big) = \lambda\big(0, \bar{a}(t)\big) + \big(0, (\lambda\dot{\bar{x}}_1(t) - q_2^x(t), \lambda\dot{\bar{x}}_2(t) - q_2^x(t))\big)$ for a.e. $t \in [0,1]$;

(6) $q^a(t) = 0$ for a.e. $t \in [0,1]$;

(7) $q^x(t) = p^x(t) - \gamma([t,1]), \quad q^a(t) = p^a(t)$ for a.e. $t \in [0,1]$;

(8) $-p^x(1) = \big(\eta_1(1), \eta_2(1)\big) \in N\big(\bar{x}(1) - \bar{u}(1); C\big)$;

(9) $\lambda + \|p^x(1)\| \neq 0$.

Employing the first condition in (4.7) together with (2) and (4), we obtain that $\dot{\bar{x}}(t) =$

$-\bar{a}(t)$ for a.e. $t \in [0,1]$. It also follows from (5)–(7) that

$$\lambda\bar{a}(t) = \lambda\dot{\bar{x}}(t) - q^x(t), \;\text{ i.e., }\; 2\lambda\bar{a}(t) = -q^x(t) \;\text{ for a.e. }\; t \in [0,1] \tag{4.8}$$

Using (5) again tells us that $p^x(\cdot)$ is constant on $[0,1]$, i.e., $p^x(t) \equiv p^x(1)$. This allows us to deduce that

$$q^x(t) = p^x(1) - \gamma([t,1]) = p^x(1) - \gamma(\{1\}) \;\text{ for a.e. }\; t \in [0,1]$$

by using the measure nonatomicity condition (a) from Theorem 3.2 for the measure γ and repeating the arguments of Example 4.1. This shows by (4.8) and the control continuity that $\bar{a}(\cdot)$ is a constant on $[0,1]$ provided that $\lambda \neq 0$; otherwise, we do not have enough information to proceed. Putting $\bar{a}(t) \equiv (\vartheta_1, \vartheta_2)$ for all $t \in [0,1]$ gives us $\bar{x}(t) = (-\vartheta_1 t, -1 - \vartheta_2 t)$ for all $t \in [0,1]$. Thus $\bar{x}(1) = (-\vartheta_1, -1 - \vartheta_2)$, and by the second condition in (4.7) we have the following two possibilities:

(a): $\bar{x}_1(1) = 1$. Then $\vartheta_1 = -1$ and the cost functional reduces is $J[\bar{x}, \bar{a}] = 1 + \vartheta_2^2$. It obviously achieves its absolute minimum value $\bar{J} = 1$ at the point $\vartheta_2 = 0$.

(b): $\bar{x}_2(1) = 0$. Then $\vartheta_2 = -1$, and the minimum cost is $\bar{J} = 1$ achieved at $\vartheta_1 = 0$.

As a result, we arrive at are two feasible solutions giving the same optimal value to the cost functionals:

$$\bar{x}(t) = (t, -1),\; \bar{a}(t) = (-1, 0) \quad\text{and}\quad \bar{x}(t) = (0, t-1),\; \bar{a}(t) = (0, -1).$$

Figure 2 provides some illustration of the sweeping motion in this case, where the red lines indicate the boundary points at which the corresponding sweeping trajectories hit the state constraints.

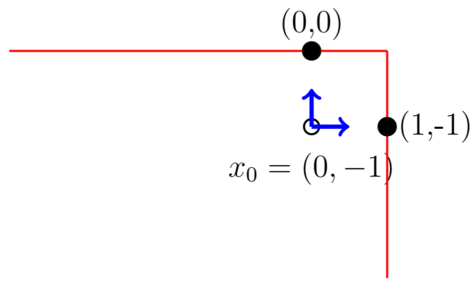

Figure 2: Two-dimensional motion.

CHAPTER 5 CONTROLLED CROWD MOTION MODEL IN A CORRIDOR

This section is devoted to the formulation and solution of an *optimal control problem* concerning the so-called *crowd motion model in a corridor*. We refer the reader to [29,30,51] for describing of the dynamics in such and related crowd motion models as a *sweeping process* with the corresponding mathematical theory, numerical simulations, and various applications. However, neither these papers nor other publications contain, to the best of our knowledge, control and/or optimization versions of crowd motion modeling, which is of our main interest here. We follow the terminology and notation of [29,30,51].

Our main goal is to demonstrate that the necessary optimality conditions obtained in Corollary 3.3 allow us to develop an effective procedure to determine optimal solutions in the general setting under consideration with finitely many participants and then explicitly solve the problem in some particular situations involving two and three participants. Furthermore, in this way we reveal certain specific features of the obtained necessary optimality conditions for problems with state constraints.

The crowd motion model of [29,30,51] is designed to deal with local interactions between participants in order to describe the dynamics of pedestrian traffic. This model rests on the following postulates:

• A spontaneous velocity corresponding to the velocity that each participant would like to have in the absence of others is defined first.

• The actual velocity is then calculated as the projection of the spontaneous velocity onto the set of admissible velocities, i.e., such velocities that do not violate certain nonoverlapping constraints.

In what follows we consider n participants ($n \geq 2$) identified with rigid disks of the same radius R in a corridor as depicted in Figure 3.

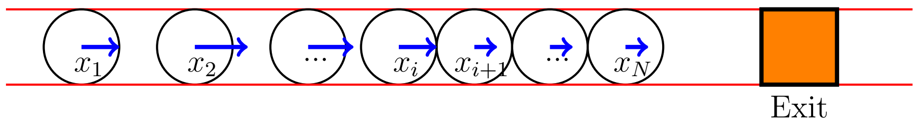

Figure 3: Crowd motion model in a corridor

In that case, since the participants are not likely to leap across each other, it is natural to restrict the set of feasible configurations to one of its connected components (*nonoverlapping condition*):

$$Q_0 = \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ x_{i+1} - x_i \geq 2R\big\}. \tag{5.1}$$

Assuming that the participants exhibit the same behavior, their *spontaneous velocity* can be written as

$$U(x) = \big(U_0(x_1), \ldots, U_0(x_n)\big) \text{ for } x \in Q_0,$$

where Q_0 is taken from (5.1). Observe that the nonoverlapping constraint in (5.1) does not allow the participants to move with their spontaneous velocity, and the distance between two participants in contact can only increase. To reflect this situation, the set of *feasible velocities*

$$C_x := \big\{v = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid x_{i+1} - x_i = 2R \Longrightarrow v_{i+1} \geq v_i \text{ for all } i = 1, \ldots, n-1\big\},$$

and then describe the *actual velocity field* is the feasible field via the Euclidean projection of $U(x)$ to C_x:

$$\dot{x}(t) = \Pi\big(U(x); C_x\big) \text{ for a.e. } t \in [0, T], \quad x(0) = x_0 \in Q_0,$$

where $T > 0$ is a fixed duration of the process and x_0 indicates the starting position of the participants. Using the orthogonal decomposition via the sum of mutually polar cone as in [30, 51], we get

$$U(x) \in N_x + \dot{x}(t) \;\text{ for a.e. }\; t \in [0,T], \quad x(0) = x_0,$$

where N_x stands for the normal cone to Q_0 at x and can be described in this case as the polar

$$N_x = C_x^* := \big\{w \in \mathbb{R}^n \mid \langle w, v\rangle \le 0 \;\text{ for all }\; v \in C_x\big\}, \quad x \in Q_0.$$

Let us now rewrite this model in the form used in our problem $(\widetilde{P})$ without control parameters so far. Given the orths $(e_1, \ldots, e_n) \in \mathbb{R}^n$, specify the polyhedral set C by

$$C := \big\{x \in \mathbb{R}^n \mid \langle x_i^*, x\rangle \le 0,\; i = 1, \ldots, n-1\big\} \;\text{ with }\; x_i^* := e_i - e_{i+1}, \quad i = 1, \ldots, n-1. \quad (5.2)$$

Since all the participants exhibit the same behavior and want to reach the exit by the shortest path, their spontaneous velocities can be represented as

$$U(x) = \big(U_0(x_1), \ldots, U_0(x_n)\big) \;\text{ with }\; U_0(x) = -s\nabla D(x),$$

where $D(x)$ stands for the distance between the position $x = (x_1, \ldots, x_n) \in Q_0$ and the exit, and where the scalar $s \ge 0$ denotes the speed. Since $x \ne 0$ and hence $\|\nabla D(x)\| = 1$, we have $s = \|U_0(x)\|$. By taking this into account as well as the aforementioned postulate that, in the absence of other participants, each participant tends to remain his/her spontaneous velocity until reaching the exit, the (uncontrolled) perturbations in this model are described by

$$f(x) = -(s_1, \dots, s_n) \in \mathbb{R}^n \text{ for all } x = (x_1, \dots, x_n) \in Q_0,$$

where s_i denotes the speed of the participant $i = 1, \dots, n$. However, if participant i is in contact with participant $i+1$ in the sense that $x_{i+1}(t) - x_i(t) = 2R$, then both of them tend to adjust their velocities in order to keep the distance at least $2R$ with the participant in contact. To control the actual speed of all the participants in the presence of the nonoverlapping condition (5.1), we suggest to involve control functions $a(\cdot) = (a_1(\cdot), \dots, a_n(\cdot))$ into perturbations as follows:

$$f\big(x(t), a(t)\big) = \big(s_1 a_1(t), \dots, s_n a_n(t)\big), \quad t \in [0, T]. \tag{5.3}$$

In order to represent this controlled crowd motion model in the form of $(\widetilde{P})$, define recurrently the vector function $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)\colon [0, T] \to \mathbb{R}^n$, which is constant in our case, by

$$\bar{u}_{i+1}(t) - \bar{u}_i(t) = 2R \text{ with } \bar{u}_1(t) = \alpha \text{ and } \|\bar{u}(t)\| = r, \quad i = 1, \dots, n-1, \tag{5.4}$$

where $r = r(\alpha)$ is an increasing function of α with the value of α specified later. Note that the nonoverlapping condition (5.1) can be written now, due to (5.2) and (5.4), via the state constraints

$$x(t) - \bar{u}(t) \in C \text{ for all } t \in [0, T], \tag{5.5}$$

where the points $t \in [0, T]$ with $x_{i+1}(t) - x_i(t) = 2R$ are exactly those at which the motion

$x(t)-\bar{u}(t)$ hits the polyhedral constraint set C.

The constructions above allow us to present the *controlled crowd motion dynamics* as

$$\begin{cases} -\dot{x}(t)\in N\big(x(t);C(t)\big)+f\big(x(t),a(t)\big) \text{ for a.e. } t\in[0,T],\\ C(t):=C+\bar{u}(t),\ \|\bar{u}(t)\|=r \text{ on } [0,T], \quad x(0)=x_0\in C(0), \end{cases} \tag{5.6}$$

with C, f, and $\bar{u}$ taken from (5.2), (5.3), and (5.4), respectively. Recall that the state constraints (5.5) are implicitly present in (5.6) due to definition (1.2) of the normal cone to convex sets.

To optimize dynamics (5.6) by using controls $a(\cdot)$, we introduce the *cost functional*

$$\text{minimize } J[x,a]:=\frac{1}{2}\Big(\|x(T)\|^2+\int_0^T\|a(t)\|^2dt\Big) \tag{5.7}$$

the meaning of which is to *minimize the distance* of all the participants to the exit at the origin together with the *energy* of feasible controls $a(\cdot)$. Having now the formulated optimal control problem for the crowd motion model in the form of $(\widetilde{P})$, we can apply to solving this problem the necessary optimality conditions for the sweeping process with controlled perturbations derived in Corollary 3.3.

It is easy to see that all the assumptions of Corollary 3.3 are satisfied for problem (5.6), (5.7). To make sure that the nontriviality condition holds in the enhanced/nondegenerate form (3.54), we select the parameter α in (5.4) so large that

$$r=r(\alpha)>l=\|x_0\|+e^{2MT}2MT\big(1+\|x_0\|\big)$$

where the number $l>0$ is calculated in (2.3) for the constant control $u(\cdot)$. As mentioned in (3.12) of Theorem 3.2, this condition with $\tau=0$ yields the validity of the second condition therein, which ensures in turn the fulfillment of the enhanced nontriviality (3.54) in

Corollary 3.3.

Applying now the necessary optimality conditions of Corollary 3.3 gives us the following, where $\lambda \geq 0$ and $\eta_i(\cdot) \in L^2([0,T];\mathbb{R}_+)$ well defined at $t = T$:

(1) $w(t) = \big(0, \bar{a}(t)\big),\ v(t) = (0,0)$ for a.e. $t \in [0,T]$;

(2) $-\dot{\bar{x}}(t) = \displaystyle\sum_{i=1}^{n-1} \eta_i(t)x_i^* + (s_1\bar{a}_1(t), \ldots, s_n\bar{a}_n(t))$ for a.e. $t \in [0,T]$;

(3) $\bar{x}_{i+1}(t) - \bar{x}_i(t) > 2R \Longrightarrow \eta_i(t) = 0$ for all $i = 1, \ldots, n-1$ and a.e. $t \in [0,T]$;

(4) $\eta_i(t) > 0 \Longrightarrow q_i^x(t) = q_{i+1}^x(t)$ for all $i = 1, \ldots, n-1$ and a.e. $t \in [0,T]$;

(5) $\dot{p}(t) = \big(0, \lambda\bar{a}(t) - (s_1 q_1^x(t), \ldots, s_n q_n^x(t))\big)$ for a.e. $t \in [0,T]$;

(6) $q^x(t) = p^x(t) - \gamma([t,T])$ for a.e. $t \in [0,T]$;

(7) $q^a(t) = p^a(t) = 0$ for a.e. $t \in [0,T]$;

(8) $-p^x(T) = \lambda\bar{x}(T) + \sum_{i \in I(\bar{x}(T)-\bar{u}(T))} \eta_i(T)$;

(9) $\sum_{i \in I(\bar{x}(T)-\bar{u}(T))} \eta_i(T) \in N\big(\bar{x}(T) - \bar{u}(T); C\big)$;

(10) $p^a(T) = 0$;

(11) $\lambda + \|q^x(0)\| + \|p^x(T)\| \neq 0$.

As discussed above, the situation where $\bar{x}_{i+1}(t_1) - \bar{x}_i(t_1) = 2R$ for some $t_1 \in [0,T]$ pushes participants i and $i+1$ to adjust their speeds in order to keep the distance at least $2R$ with the one in contact. It is natural to suppose that both participants i and $i+1$ maintain their new constant velocities after the time $t = t_1$ until either reaching someone ahead or the end of the process at time $t = T$. Hence the velocities of all the participants are piecewise constant on $[0,T]$ in this setting.

Observe that the differential relation in (2) can be read as

$$\begin{cases} -\dot{\bar{x}}_1(t) = \eta_1(t) + s_1\bar{a}_1(t), \\ -\dot{\bar{x}}_i(t) = \eta_i(t) - \eta_{i-1}(t) + s_i\bar{a}_i(t), \quad i = 2,\ldots,n-1, \\ -\dot{\bar{x}}_n(t) = -\eta_{n-1}(t) + s_n\bar{a}_n(t) \end{cases} \tag{5.8}$$

for a.e. $t \in [0,T]$. Next we clarify the sense of the implications in (3). If participant 1 is far away from participant 2 in the sense that $\bar{x}_2(t) - \bar{x}_1(t) > 2R$ for some time $t \in [0,T]$, then his/her actual velocity and the spontaneous velocity are the same meaning that $-\dot{\bar{x}}_1(t) = s_1\bar{a}_1(t)$. Likewise we have the same situation for the last participant n. However, it is not the case for two adjacent participants between the first and last ones because they must rely on the participants ahead and behind them.

Further, it follows from conditions (5) and (7) that

$$\lambda\bar{a}_i(t) = s_i q_i^x(t) \text{ for a.e. } t \in [0,T] \text{ and all } i = 1,\ldots,n. \tag{5.9}$$

If $\eta_i(t) > 0$ for some $i \in \{1,\ldots,n-1\}$ and $t \in [0,T]$, we deduce from (4) and (5.9) by taking into account the continuity of $\bar{a}_i(\cdot)$ on $[0,T]$ that

$$s_{i+1}\bar{a}_i(t) = s_i\bar{a}_{i+1}(t) \text{ for all } t \in [0,T] \text{ and } i = 1,\ldots,n-1 \tag{5.10}$$

provided that $\lambda > 0$ (say $\lambda = 1$); otherwise, we do not have enough information to proceed. Since the velocities s_i are constant in (5.11), it is to suppose by (5.10) that the functions $\bar{a}_i(\cdot)$ are constant $\bar{a}_i$ on $[0,T]$ for all $i = 1,\ldots,n$ and thus optimal controls among such functions. Using this and the Newton-Leibniz formula in (5.8) gives us the trajectory representations for all $t \in [0, T]$:

$$\begin{cases} \bar{x}_1(t) = x_{01} - \displaystyle\int_0^t \eta_1(s)ds - ts_1\bar{a}_1, \\ \bar{x}_i(t) = x_{0i} + \displaystyle\int_0^t \big[\eta_{i-1}(s) - \eta_i(s)\big]ds - ts_i\bar{a}_i \ \text{ for } \ i = 2, \ldots, n-1, \\ x_n(t) = x_{0n} + \displaystyle\int_0^t \eta_{n-1}(s)ds - ts_n\bar{a}_n, \end{cases} \tag{5.11}$$

where $(x_{01}, \ldots, x_{0n})$ are the components of the starting point $x_0 \in \mathbb{R}^n$ in (5.6).

Prior to developing an effective procedure to find optimal solutions to the controlled crowd motion model by using the obtained optimality conditions in the general case above, we consider the following example for two participants that shows how to explicitly solve the problem in such settings.

Example 5.1 (solving the crowd motion control problem with two participants). Specify the data of (5.6), (5.7) as follows: $n = 2,\ T = 6,\ s_1 = 6,\ s_2 = 3,\ x_{01} = -60,\ x_{02} = -48,\ R = 3$. Then the equations in (5.11) reduce for all $t \in [0, 6]$ to

$$\bar{x}_1(t) = -60 - \int_0^t \eta(s)ds - 6t\bar{a}_1, \quad \bar{x}_2(t) = -48 + \int_0^t \eta(s)ds - 3t\bar{a}_2. \tag{5.12}$$

Let $t_1 \in [0, 6]$ be the first time when $\bar{x}_2(t_1) - \bar{x}_1(t_1) = 2R = 6$; see Figure 4.

Figure 4: Two participants out of contact for $t < t_1$.

Hence for $t < t_1$ we have $x_2(t) - x_1(t) > 2R = 6$, and so $\eta(t) = 0$ by (3). Note that at the point $t = t_1$ the motion $\bar{x}(t) - \bar{u}(t)$ hits the state constraint set C in (5.5) and thus is reflected by a nonzero measure γ in (6). However, we can proceed by an easier way in our particular setting. Indeed, subtracting the first equation in (5.12) from the second one gives us the relationship

$$12 - 3t_1(\bar{a}_2 - 2\bar{a}_1) = 6 \text{ and thus } 6\bar{a}_1 - 3\bar{a}_2 + 1 \le 0. \tag{5.13}$$

Suppose without loss of generality that both functions $\eta(t)$ and $\dot{\bar{x}}(t_1)$ are well defined at $t = t_1$. Then we get from (5.8) and (5.12) in this case the expressions

$$\dot{\bar{x}}_1(t_1) = -\eta(t_1) - 6\bar{a}_1 \text{ and } \dot{\bar{x}}_2(t_1) = \eta(t_1) - 3\bar{a}_2$$

with $\dot{\bar{x}}_1(t_1) \le \dot{\bar{x}}_2(t_1)$, which imply in turn that

$$-2\eta(t_1) - 6\bar{a}_1 + 3\bar{a}_2 \le 0. \tag{5.14}$$

It follows from (5.13) and (5.14) that $\eta(t_1) \ge 1/2$. Furthermore, we deduce from (5.10) with the chosen speed values s_1, s_2 that the constant controls $\bar{a}_1, \bar{a}_2$ are related by

$$\bar{a}_1 = \frac{s_1}{s_2}\bar{a}_2 = 2\bar{a}_2.$$

Having in hand the relationships above, let us now calculate an optimal solution to the problem under consideration by imposing the requirement that both participants maintain their new constant velocities until the end of the process at $t = T$, i.e., satisfying the condition $\dot{\bar{x}}(t) = \dot{\bar{x}}(t_1)$ for all $t \in [t_1, 6]$. Since $\bar{a}(\cdot)$ is constant on $[0, 6]$ and $\dot{\bar{x}}(\cdot)$ is constant on the intervals $[0, t_1)$ and $[t_1, 6]$, the vector function $\eta(\cdot)$ is constant on $[0, t_1)$ and $[t_1, 6]$ while admitting by (5.11) the representation

$$\eta(t) = \begin{cases} \eta(0) & \text{a.e. } t \in [0, t_1) \text{ including } t = 0, \\ \eta(t_1) & \text{a.e. } t \in [t_1, 6] \text{ including } t = t_1. \end{cases}$$

In particular, $\eta(t) = \eta(t_1) > 0$ a.e. on $[t_1, 6]$, and thus we get from (3) that $\bar{x}_2(t) - \bar{x}_1(t) = 2R = 6$ for all $t \in [t_1, 6]$, i.e., the optimal motion stays on the boundary of state constraints

(5.5) on the whole interval $[t_1, 6]$ meaning that the two participants of the model are in contact on this interval; see Figure 5.

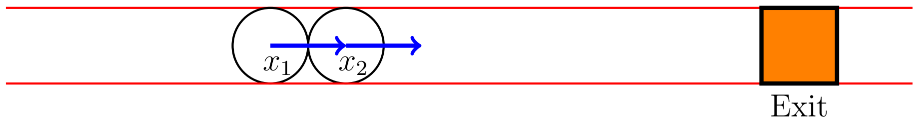

Figure 5: Two participants in contact for $t \geq t_1$.

Combining this with the the subtraction of the first equation from the second one in (5.12) gives us

$$(t - t_1)\big[2\eta(t_1) + 6\bar{a}_1 - 3\bar{a}_2\big] = 0 \ \text{ for all } t \in [t_1, 6],$$

which in turn implies that $2\eta(t_1) + 6\bar{a}_1 - 3\bar{a}_2 = 0$. Remembering that $\bar{a}_1 = 2\bar{a}_2$, we calculate the value of $\eta(\cdot)$ at the hitting point $t = t_1$ by $\eta(t_1) = -\frac{9}{2}\bar{a}_2 = -\frac{9}{4}\bar{a}_1$. Note also that $\dot{\bar{x}}_2(t_1) = \dot{\bar{x}}_1(t_1)$ in our case. Based on these calculations, we can express the value of cost functional (5.7) for this example at $(\bar{x}, \bar{a})$ as

$$J[\bar{x}, \bar{a}] = \frac{1}{2}\Big[\big(45\bar{a}_2 + 57\big)^2 + \big(45\bar{a}_2 + 51\big)^2\Big] + 15\bar{a}_2^2.$$

Minimizing this function of $\bar{a}_2$ subject to the constraint $\bar{a}_2 \leq -\frac{1}{9}$ that comes from the second expression in (5.13) gives us the optimal control value $\bar{a}_2 = -\frac{4860}{4080} \approx -1.1911$. Accordingly the formulas obtained above allows us to calculate all the other ingredients of the optimal solution with the corresponding values of dual variables in the necessary optimality condition. It gives us, in particular, that

$$\gamma\big([t, 6]\big) \approx (-1.56, 3.76) \ \text{ for } \ 0.56 = t_1 \leq t \leq 6,$$

which reflects the fact that the optimal sweeping motion hits the boundary of the state constraints at $t_1 = 0.56$ and stays there till the end of the process at $T = 6$. It is worth mentioning that the obtained nonzero measure γ has the opposite signs of its components on $[t_1, 6]$, which is different from the standard optimal control problems with inequality state constraints.

Now we come back to the *general case* of the controlled crowd model in a corridor with $n \geq 3$ participants. Following the approach employed in Example 5.1, we develop an *effective procedure* to determine an optimal control from the obtained necessary optimality conditions and then fully implement by a numerical example for the case where $n = 3$.

Recall our postulate that any two adjacent participants i and $i+1$ that come to be in contact at some point $t \in [0, T]$ (i.e., $x_{i+1}(t) - x_i(t) = 2R$) have the same velocity therein, change their velocities at the contact point, and maintain their new constant velocities until reaching the participant ahead or until the end of the process at $t = T$. This yields that the function $\eta(\cdot)$ in the conditions above is piecewise constant on $[0, T]$. Suppose for simplicity that $\eta_0(t) = \eta_n(t) = 0$ on $[0, T]$ and then rewrite (5.11) as

$$\bar{x}_i(t) = x_{0i} + \int_0^t \big[\eta_{i-1}(s) - \eta_i(s)\big] ds - ts_i\bar{a}_i \quad \text{for} \quad i = 1, \ldots, n.$$

Fix $i \in \{1, \ldots, n-1\}$, and let t_i be the first time when $\bar{x}_{i+1}(t_i) - \bar{x}_i(t_i) = 2R$; see Figure 6.

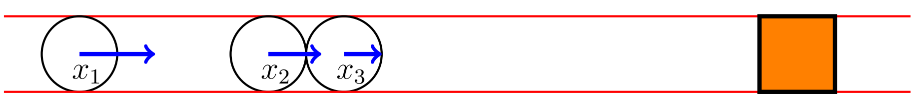

Figure 6: Out of contact situation for two adjacent participants when $t < t_1$

For each such index i consider the numbers

$$\vartheta^i := \min\big\{t_j \big| \, t_j > t_i, \; j = 1, \ldots, n-1\big\}, \quad \vartheta_i := \max\big\{t_j \big| \, t_j < t_i, \; j = 1, \ldots, n-1\big\} \quad (5.15)$$

and observe the following relationships for the optimal crowd motion on the intervals $[0, t_i)$

and $\in [t_i, \vartheta^i)$:

- If $t \in [0, t_i)$, we have $\eta_i(\cdot) = 0$ on this interval by (3). This gives us

$$\bar{x}_i(t) = x_{0i} + \int_0^t \eta_{i-1}(s)ds - ts_i\bar{a}_i, \quad \bar{x}_{i+1}(t) = x_{0(i+1)} - \int_0^t \eta_{i+1}(s)ds - ts_{i+1}\bar{a}_{i+1} \ \text{ for } \ t \in [0, t_i).$$

- If $t \in [t_i, \vartheta^i)$ with ϑ^i from (5.15), we have on this interval that

$$\begin{cases} \bar{x}_i(t) = x_{0i} + \displaystyle\int_0^{t_i} \eta_{i-1}(s)ds + (t - t_i)\big[\eta_{i-1}(t_i) - \eta_i(t_i)\big] - ts_i\bar{a}_i, \\ x_{i+1}(t) = x_{0(i+1)} - \displaystyle\int_0^{t_i} \eta_{i+1}(s)ds + (t - t_i)\big[\eta_i(t_i) - \eta_{i+1}(t_i)\big] - ts_{i+1}\bar{a}_{i+1}. \end{cases}$$

In what follows we suppose without loss of generality that the functions $\dot{\bar{x}}(\cdot)$ are well defined at t_i while the functions $\eta(\cdot)$ are well defined at t_i and ϑ_i. Since at the contact time $t = t_i$ the distance between the two participants i and $i+1$ is exactly $2R$ (see Figure 7), we have the following relationships:

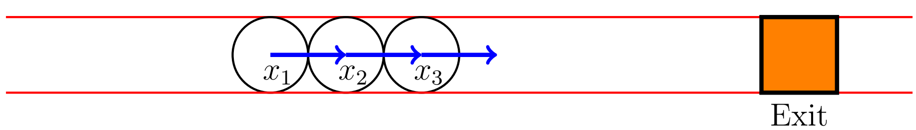

Figure 7 All the participants in contact for $t \geq t_1$

$$2R = \bar{x}_{i+1}(t_i) - \bar{x}_i(t_i) = x_{0,(i+1)} - x_{0i} - \int_0^{t_i} \big[\eta_{i+1}(s) + \eta_{i-1}(s)\big]ds - t_i\big(s_{i+1}\bar{a}_{i+1} - s_i\bar{a}_i\big)$$

$$= x_{0(i+1)} - x_{i0} - \int_0^{\vartheta_i} \big[\eta_{i+1}(s) + \eta_{i-1}(s)\big]ds - (t_i - \vartheta_i)\big[\eta_{i+1}(\vartheta_i) + \eta_{i-1}(\vartheta_i)\big]$$

$$- t_i(s_{i+1}\bar{a}_{i+1} - s_i a_i),$$

where ϑ_i is defined in (5.15) being dependent of t_i. Then we can find $t_i \leq T$ from the equation

$$t_i = \frac{x_{0(i+1)} - x_{0i} - 2R + \vartheta_i\big[\eta_{i+1}(\vartheta_i) + \eta_{i-1}(\vartheta_i)\big] - \displaystyle\int_0^{\vartheta_i} \big[\eta_{i+1}(s) + \eta_{i-1}(s)\big]ds}{\eta_{i+1}(\vartheta_i) + \eta_{i-1}(\vartheta_i) + s_{i+1}\bar{a}_{i+1} - s_i\bar{a}_i} \tag{5.16}$$

provided that $x_{0(i+1)} - x_{0i} > 2R$. In the case where $x_{0(i+1)} - x_{0i} = 2R$ we put $t_i = 0$. Our postulate tells us that $\dot{x}_{i+1}(t_i) = \dot{x}_i(t_i)$, which implies therefore that

$$2\eta_i(t_i) = \eta_{i+1}(t_i) + \eta_{i-1}(t_i) + s_{i+1}\bar{a}_{i+1} - s_i\bar{a}_i. \tag{5.17}$$

If $\eta_i(t_i) > 0$, we get from the above that (5.10) holds, while the remaining case where $\eta_i(t_i) = 0$ can be treated via (5.17). The cost functional (5.7) can be expressed in this way as a function of $(\bar{a}_1, \ldots, \bar{a}_n)$ and $\eta_i(t_j)$ for $i = 0, \ldots, n$ and $j = 1, \ldots, n-1$. Consequently the optimal control problem under consideration reduces to the finite-dimensional optimization of this cost subject to equality (5.16) and (5.17) constraints. To furnish these operations step-by-step, we proceed as follows:

Step 1: Determine which participants are in contact at the initial time, i.e., for which $i \in \{1, \ldots, n\}$ we have $x_{0(i+1)} - x_{0i} = 2R$. If this occurs only for $i = n$, there is nothing to do. If it is the case of some $i \in \{0, \ldots, n-1\}$, we put $t_i := 0$ and observe that participants i and $i+1$ have the same velocities while being away by $2R$ from each other.

Step 2: If $x_{0(i+1)} - x_{0i} = 2R$ for all $i = 0, \ldots, n-1$, express t_i as a function of $\bar{a}_i$ and η_i by solving equation (5.16) for t_i with ϑ_i taken from in (5.15).

Step 3: Find relations between η_i and $\bar{a}_i$ from (5.10) and (5.17), respectively, and substitute them into the cost function (5.7) for the subsequent optimization with respect to $\bar{a}_i$.

We now demonstrate how this procedure works in the case where $n = 3$ in the crowd motion model.

Example 5.2 (solving the crowd motion control problem with three participants). Consider the optimal control problem in (5.6), (5.7) with the following initial data:

$$n = 3,\ s_1 = 6,\ s_2 = 3,\ s_3 = 2,\ x_{01} = -60,\ x_{02} = -48,\ x_{03} = -42,\ T = 6,\ R = 3.$$

By using the procedure outlined above, we first get $x_{02} - x_{01} = 12 > 6 = 2R$ and $x_{03} - x_{02} = 6 = 2R$. Then it is obvious that $t_2 = 0$, t_1 is determined by (5.16) as

$$t_1 = \frac{6}{\eta_2(0) + 3\bar{a}_2 - 6\bar{a}_1} \leq 6,$$

and thus $\vartheta_1 = t_2 = 0$. It is easy to see that in this example we have

$$\dot{\bar{x}}_1(t) = -6\bar{a}_1,\ \dot{\bar{x}}_2(t) = -\eta_2(0) - 3\bar{a}_2,\ \dot{\bar{x}}_3(t) = \eta_2(0) - 2\bar{a}_3,$$

$$\bar{x}_1(t) = -60 - 6\bar{a}_1,\ \bar{x}_2(t) = -48 - t\eta_2(0) - 3t\bar{a}_2,\ \bar{x}_3(t) = -42 + t\eta_2(0) - 2t\bar{a}_3$$

for $0 \leq t < t_1$, while for $t \in [t_1, 6]$ the corresponding formulas are:

$$\dot{\bar{x}}_1(t) = -\eta_1(t_1) - 6\bar{a}_1,\ \dot{\bar{x}}_2(t) = -\eta_2(t_1) + \eta_1(t_1) - 3\bar{a}_2,\ \dot{\bar{x}}_3(t) = \eta_2(t_1) - 2\bar{a}_3,$$

$$\begin{cases} \bar{x}_1(t) = -60 - (t - t_1)\eta_1(t_1) - 6t\bar{a}_1,\ \bar{x}_2(t) = -48 - t\eta_2(0) + (t - t_1)\big(\eta_1(t_1) - \eta_2(t_1)\big) - 3t\bar{a}_2, \\ \bar{x}_3(t) = -42 + t\eta_2(0) + (t - t_1)\eta_2(t_1) - 2t\bar{a}_3. \end{cases}$$

It follows directly from (5.17) the following relationships for $\eta(\cdot)$:

$$2\eta_1(t_1) = \eta_2(t_1) + 3\bar{a}_2 - 6\bar{a}_1,\ 2\eta_2(0) = 2\bar{a}_3 - 3\bar{a}_2,\ 2\eta_2(t_1) = \eta_1(t_1) + 2\bar{a}_3 - 3\bar{a}_2.$$

Denoting for convenience $x := a_2,\ y := a_3,\ z := a_1$ and taking into account that $\bar{a}_1 = 2\bar{a}_2$ by (5.10) due to $\eta_1(t_1) > 0$, we rewrite these expressions and the above formula for t_1 as

$$t_1 = \frac{6}{-(21/2)x + y},\ \eta_1(t_1) = -8x + (13/6)y,\ \eta_2(0) = -(3/2)x + y,\ \eta_2(t_1) = -3x + (4/3)y. \tag{5.18}$$

Let us split the situation into the following two cases:

Case 1: $\eta_2(0) > 0$. In this case we have $x = \frac{3}{2}y$, and thus (5.18) gives us the calculations:

$$t_1 = -(24/59y) \leq 6,\ \eta_1(t_1) = -(59/6)y,\ \eta_2(0) = -(5/4)y,\ \eta_2(t_1) = -(37/6)y.$$

As a result, we have the expressions for the terminal points of the optimal trajectories

$$\bar{x}_1(6) = -49y - 56,\ \bar{x}_2(6) = -49y - 50,\ \bar{x}_3(6) = -49y - 44$$

and the corresponding representation of the cost function

$$J = \frac{1}{2}\Big[(49y+56)^2 + (49y+50)^2 + (49y+44)^2\Big] + 36.75y^2.$$

Minimizing this quadratic function over the constraint $y \leq -\frac{4}{59}$ lead us to the optimal point $y = -\frac{7350}{7276.5} \approx -1.01$ and the corresponding values of the optimal control $\bar{a}(t) = (-3.03, -1.52, -1.01)$, which creates the optimal contact time $t_1 = 0.40$ and the optimal crowd motion dynamics

$$\big(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)\big) = \begin{cases} (18.18t - 60,\ 3.28t - 48,\ 3.28t - 42) & \text{for } t \in [0, t_1), \\ (8.25t - 56,\ 8.25t - 50,\ 8.25t - 44) & \text{for } t \in [t_1, 6]. \end{cases}$$

Note also that $\gamma([t, 6]) = (-2.92,\ 4.71,\ 1.24)$ when $t \in [t_1, 6]$ with $\lambda = 1$ as considered above.

Case 2: $\eta_2(0) = 0$. Then we can deduce from (5.18) that

$$t_1 = -y^{-1} \leq 6,\ \eta_1(t_1) = -(19/6)y,\ \eta_2(t_1) = -(2/3)y,\ \eta_2(0) = -(3/2)x + y = 0.$$

Hence $\eta_2(t_1) > 0$, which implies by (5.10) that $2\bar{a}_2 = 3\bar{a}_3$ and so $x = \frac{3}{2}y$. Combining the latter with the above relation $x = \frac{2}{3}y$ tells us that $x = y = 0$ This contradicts the constraint $y < 0$ and thus rules out the situation in case. Overall, the calculations in Case 1 completely solve the crowd motion optimal control problem in this example by using the optimality conditions established in Corollary 3.3.

CHAPTER 6 CONCLUDING REMARKS AND FUTURE DIRECTIONS

In the future, we would like to continue our research on the following issues.

- **Optimal Control of a Perturbed Sweeping Process and Applications** Currently, optimality conditions for the optimal control problems associated with *perturbed sweeping process* have been derived successfully (see our paper [10]). Nevertheless, the structure of the moving set $C(t) = C + u(t)$ in (2.6), as a translation of the convex polyhedral set, considered in the sweeping process problem is somehow restrictive. It may cause the restriction of the applications to the crowd motion models in the sense that: we just considered only the case when n individuals move in a corridor. Following the work of J. Venel in [51], in order to fit higher dimensional motions, we will consider a more general form of the moving set $C(t)$, possibly

$$C(t) := \{x \in H \mid g_i(t, x) \geq 0, \text{ for } i = 1, ..., m\},$$

  where g_i are convex functions. We then try to derive necessary optimality conditions for local minimizers of the new controlled sweeping process with the set $C(t)$ specified above using the method of discrete approximations. Actually, optimality conditions have been obtained successfully for discrete optimal solutions entirely via the initial data of the perturbed sweeping process under consideration in our ongoing research. A challenging issue remains on deriving necessary optimality conditions for local solutions to continuous-time sweeping control problems of this class by passing to the limit from those obtained for their finite-difference counterparts. Besides their own theoretical interest, explicit necessary optimality conditions for continuous-time sweeping systems may be convenient for calculating optimal solutions. We pursue these goals in both theory and applications, particularly to the crowd motion model in a more general setting and to a well-known game (the labyrinth table-top game) in our on-going research.

- **Optimal Control of differential inclusions under some weaker assumptions than Lipschitz continuity imposed on the velocity set F.** We will consider the generalized Bolza problem governed by differential inclusions

$$\dot{x}(t) \in F(x(t), t) \quad \text{a.e} \quad t \in [a, b] \quad \text{with} \quad x(a) = x_0,$$

where the set-valued mapping F satisfies some weaker condition than Lipschitzian property, e.g., one-sided Lipschitzian (OSL), relaxed one-sided Lipschitzian (ROSL), modified one-sided Lipschitzian (MOSL) (see [19, 38] for more discussions and references therein), subject to some endpoint constraints. In fact, the *velocity set* $F(x(t), x) = -N(x(t); C(t))$ in the sweeping process which is not Lipschitzian actually satifies the OSL condition. Due to the non-Lipschitzian property of F, the classical discrete approximation approach cannot be applied and does require *serious modifications*. Even in the particular case of a sweeping process, although the limiting procedure to derive optimality conditions has been obtained sucessfully, the procedure is still very complicated and it is not clear whether we can succeed in more general situations. In the paper [38], Mordukhovich and Tian can derive necessary optimality conditions for the discretized Bolza problems via suitable generalized differential constructions of variational analysis when the velocity set F satisfies ROSL condition. The obtained results on the well-posedness of discrete approximations and necessary optimality conditions allow us to justify a numerical approach to solve the generalized Bolza problem for OSL differential inclusions by using discrete approximations constructed via the implicit Euler scheme. However, to obtain optimal conditions for the original problem is still not clear and has not been done yet. It seems to us that the condition like OSL may be not adequate in order to achieve our goal and it should be modified in some way. To proceed, we may go back to the sweeping process problem to analyze what could be missing in the general problems. This will be a part of our future research.

REFERENCES

[1] L. Adam, J. V. Outrata. On optimal control of a sweeping process coupled with an ordinary differential equation. *Discrete Contin. Dyn. Syst.–Ser. B*, 19:2709–2738, 2014.

[2] S. Adly, T. Haddad, L. Thibault. Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities. *Math. Program.*, 148:5–47, 2014.

[3] H. Attouch, G. Buttazzo, G. Michaille. *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization,* SIAM, Philadelphia, 2005.

[4] A. V. Arutyunov, S. M. Assev. Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints, *SIAM J. Control Optim.,* **35** (1997), pp. 930-952.

[5] A. V. Arutyunov, D. Yu. Karamzin. Nondegenerate necessary optimality conditions for the optimal control problems with equality type state constraints. To appear in *J. Global Optim.*, DOI:10.1007/s10898-015-0272-9, 2015.

[6] J. M. Borwein and Q. J. Zhu. *Techniques of Variational Analysis.* Springer, 2005.

[7] M. Brokate, J. Sprekels. *Hysteresis and Phase Transitions.* Springer, 1996.

[8] M. Brokate, P. Krejčí. Optimal control of ODE systems involving a rate independent variational inequality. *Discrete Contin. Dyn. Syst.–Ser. B*, 18:331–348, 2013.

[9] T. H. Cao, B. S. Mordukhovich. Optimal control of a perturbed sweeping process via discrete approximations, submitted for publication; available at http://arxiv.org/abs/1511.08922.

[10] T. H. Cao, B. S. Mordukhovich. Optimality conditions for a perturbed controlled sweeping process with applications to the crowd motion model, submitted for publication; available at http://arxiv.org/abs/1511.08923.

[11] G. Colombo, L. Thibault. Prox-regular sets and applications. In Y. Gao and D. Motreanu, editors, *Handbook of Nonconvex Analysis*, pages 99–182. International Press, 2010.

[12] G. Colombo, R. Henrion, N. D. Hoang, B. S. Mordukhovich. Optimal control of the sweeping process. *Dyn. Contin. Discrete Impuls. Syst.–Ser. B*, 19:117–159, 2012.

[13] G. Colombo, R. Henrion, N. D. Hoang, B. S. Mordukhovich. Optimal control of the sweeping process over polyhedral controlled sets. To appear in *J. Diff. Eqs.*, http://dx.doi.org/10.1016/j.jde.2015.10.039, 2015.

[14] F. H. Clarke. Optimal solutions to differential inclusions, *J. Optimization Theory Appl.* **19** (1976), 469-478.

[15] F. H. Clarke. The generalized problem of Bolza, *SIAM J. Control Optim.* **14** (1976), 682-699.

[16] F. H. Clarke. *Optimization and Nonsmooth Optimization.* Wiley-Interscience, New York, 1983.

[17] F. H. Clarke. *Optimization and Nonsmooth Optimization*, Wiley Interscience, New York, 1983; reprinted as vol. 5 of Classical in Applied Mathematics, SIAM, Philadelphia, 1990.

[18] F. H. Clarke, Yu. S Ledyaev, R. J. Stern, P. R. Wolenski. *Nonsmooth Analysis and Control Theory.* Springer, 1998.

[19] T. Donchev, E. Farkhi, B. S. Mordukhovich. Discrete approximations, relaxation, and optimization of one-sided Lipschitzian differential inclusions in Hilbert spaces. *J. Diff. Eq.* **243**, pp.I301-328, (2007).

[20] J. F. Edmond, L. Thibault. Relaxation of an optimal control problem involving a perturbed sweeping process. *Math. Program.*, 104:347–373, 2005.

[21] H. Frankowska. The maximum principle for an optimal solution to a differential inclusion with endpoint constraints, *SIAM J. Control Optim.* 25 (1987), 145-157.

[22] R. Henrion, B. S. Mordukhovich, N. M. Nam. Second-order analysis of polyhedral systems in finite and infinite dimensions with applications to robust stability of variational inequalities. *SIAM J. Optim.* **20** (2010), no. 5, 2199 2227. 49K40 (49J53 58C20).

[23] M. A. Krasnosel'skiǐ, A. V. Pokrovskiǐ. *Systems with Hysteresis.* Springer, 1989.

[24] P. Krečí. Evolution variational inequalities and multidimensional hysteresis operators. In P. Drabek, P. Krečí and P. Takac, editors, *Nonlinear Differential Equations*, Res. Notes Math. 404, pages 47–110. Chapman & Hall, CRC, 1999.

[25] M. Kunze, M. D. P. Monteiro Marques. An introduction to Moreau's sweeping process. In B. Brogliato, editor, *Impacts in Mechanical Systems*, Lecture Notes in Phys. 551, pages 1–60. Springer, 2000.

[26] P. D. Loewen. *Optimal Control via Nonsmooth Analysis.* CRM Proceedings and Lecture Notes, Vol. 2 American Mathematical Society, Providence, RI, 1993.

[27] P. D. Loewen, R. T. Rockafellar. Optimal control of unbounded differential inclusions, *SIAM J. Control Optim.* **34** (1994), 442-470.

[28] Markus Kunze, D. P. Manuel, Monteiro Marques. *An Introduction to Moreau's Sweeping Process.*

[29] B. Maury, J. Venel. A mathematical framework for a crowd motion model. *C. R. Acad. Sci. Paris Ser. I*, 346:1245–1250, 2008.

[30] B. Maury, J. Venel. Handling of contacts in crowd motion simulations, *Traffic and Granular Flow '07*, Springer (2009) 171-180.

[31] B. Maury, J. Venel. A discrete contact model for crowd motion, *ESAIM: M2AN* **45** (2011) no. 1, 145-168.

[32] A. Mielke. Evolution of rate-independent systems. Evolutionary equations. Vol. II. In C. M. Dafermos and E. Feireis, editors, *Handbook of Differential Equations.* Elsevier, 2005.

[33] B. S. Mordukhovich. On variational analysis of differential inclusions, in Optimization and Nonlinear Analysis (A. Ioffe, M. Marcus, and S. Reich, eds.), *Pitman Res. Notes Math. Series* **244**, Longman, Harlow, Essex, (1992), 199-214.

[34] B. S. Mordukhovich. Sensitivity analysis in nonsmooth optimization, in Theoretical Aspects of Industrial Design, edited by D. A. Field and V. Komkov, *SIAM Proc. Appl. Math.,* **58** (1992), 32-46, Philadelphia, Pennsylvania.

[35] B. S. Mordukhovich. Discrete approximations and refined Euler-Lagrange conditions for differential inclusions. *SIAM J. Control Optim.*, 33:882–915, 1995.

[36] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, I: Basic Theory.* Springer-Verlag, Berlin (2006).

[37] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, II: Applications*. Springer-Verlag, Berlin (2006).

[38] B. S. Mordukhovich, Y. Tian. Implicit Euler approximation and optimization of one-sided Lipschitzian differential inclusions, to appear in Nonlinear Analysis and Optimization, Contemporary Mathematics Series, AMS, 2015, http://arxiv.org/abs/1410.2207.

[39] M. D. P. Monteiro Marques. *Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction*, Birkhäuser, Boston, 1993.

[40] J. J. Moreau. On unilateral constraints, friction and plasticity. In G. Capriz and G. Stampacchia, editors, *New Variational Techniques in Mathematical Physics*, Proceedings of C.I.M.E. Summer Schools, pages 173–322. Cremonese, 1974.

[41] J. J. Moreau. Application of convex analysis to the treatment of elastoplastic system, in *Applications of Methods of Functional Analysis to Problems of Mechanics (Joint Symposium IUTAM/IMU, Marseille 1975)* (P. Germain and B. Nayroles Eds.) Lecture Notes in Math. 503, Springer Verlag, 1976, pp. 56-89.

[42] J. J. Moreau. Evolution problem associated with a moving convex set in a Hilbert space, *J. Diff. Eqs.* **26** (1977), 347-374.

[43] J. S. Pang, D. E. Stewart. Differential variational inequalities. *Math. Program.*, 113:345–434, 2008.

[44] F. Rampazzo, R. Vinter. Degenerate optimal control problems with state constraints. em SIAM J. Control, Vol. **39**, No. 4, pp. 989-1007.

[45] R. T. Rockafellar, R. J-B. Wets. *Variational Analysis*. Springer-Verlag, Berlin (2004).

[46] H. Schättler, U. Ledzewicz. *Geometric Optimal Control, Theory, Methods and Examples*, Springer.

[47] A. H. Siddiqi, P. Manchanda, M. Brokate. On some recent developments concerning Moreau's sweeping process. In A. H. Siddiqi and M. Kočvara, editors, *Trends in Industrial and Applied Mathematics*, pages 339–354. Kluwer, 2002.

[48] D. E. Stewart. *Dynamics with Inequalities: Impacts and Hard Constraints*. SIAM, 2011.

[49] A. A. Tolstonogov. *Differential Inclusions in a Banach Space*. Kluwer, 2000.

[50] J. Venel. Integrating strategies in numerical modeling of crowd motion. *Pedestrian and Evacuation Dynamics '08*, Springer (2010) 641-646.

[51] J. Venel. A numerical scheme for a class of sweeping process. *Numerische Mathematik*, 118:451-484, 2011.

[52] R. B. Vinter. *Optimal Control*, Birkhauser, Boston, 2000.

ABSTRACT

OPTIMAL CONTROL OF A PERTURBED SWEEPING PROCESS WITH APPLICATIONS TO THE CROWD MOTION MODEL

by

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August 2016

Advisor: Dr. Boris. S. Mordukhovich

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

The dissertation is devoted to the study and applications of a new class of optimal control problems governed by a perturbed sweeping process of the hysteresis type with control functions acting in both play-and-stop operator and additive perturbations. Such control problems can be reduced to optimization of discontinuous and unbounded differential inclusions with pointwise state constraints, which are immensely challenging in control theory and prevent employing conventional variation techniques to derive necessary optimality conditions. We develop the method of discrete approximations married with appropriate generalized differential tools of modern variational analysis to overcome principal difficulties in passing to the limit from optimality conditions for finite-difference systems. This approach leads us to nondegenerate necessary conditions for local minimizers of the controlled sweeping process expressed entirely via the problem data. Besides illustrative examples, we apply the obtained

results to an optimal control problem associated with of the crowd motion model of traffic flow in a corridor, which is formulated in this thesis. The derived optimality conditions allow us to develop an effective procedure to solve this problem in a general setting and completely calculate optimal solutions in particular situations.

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T. H. Cao and B. S. Mordukhovich. Optimal control of a perturbed sweeping process via discrete approximations, to appear in Disc. Cont. Dyn. Syst., Ser B; `http://arxiv.org/abs/1511.08922`.